CLEAR VISIBILITY AND THE DIMENSION OF KERNELS OF STARSHAPED SETS

Marilyn Breen

ABSTRACT. This paper will use the concept of clearly visible to obtain a Krasnosel'skii-type theorem for the dimension of the kernel of a starshaped set, and the following result will be proved: For each \( k \) and \( n \), \( 1 \leq k \leq n \), let 
\[
\begin{align*}
  f(n, n) &= n + 1 \\
  f(n, k) &= 2n 
\end{align*}
\]
if \( 1 \leq k \leq n - 1 \). Let \( S \) be a nonempty compact set in \( \mathbb{R}^n \). Then for a \( k \) with \( 1 \leq k \leq n \), \( \dim \ker S \geq k \) if and only if every \( f(n, k) \) points of \( \text{bdry} \ S \) are clearly visible from a common \( k \)-dimensional subset of \( S \). If \( k = 1 \) or \( k = n \), the result is best possible. Moreover, if \( S \) is a compact, connected, nonconvex set in \( \mathbb{R}^2 \), then \( \text{bdry} \ S \) may be replaced by \( \text{lnk} \) \( S \) in the theorem.

1. Introduction. We begin with some definitions from [1]. Let \( S \) be a compact set in \( \mathbb{R}^n \). A point \( s \) in \( S \) is said to be a point of local convexity of \( S \) if and only if there is some neighborhood \( N \) of \( s \) such that \( N \cap S \) is convex. If \( S \) fails to be locally convex at \( q \) in \( S \), then \( q \) is called a point of local nonconvexity (lnc point) of \( S \). For points \( x \) and \( y \) in \( S \), we say \( x \) sees \( y \) via \( S \) (\( x \) is visible from \( y \) via \( S \)) if and only if the segment \( [x, y] \) lies in \( S \). Similarly, \( x \) is clearly visible from \( y \) via \( S \) if and only if there is some neighborhood \( N \) of \( x \) such that \( y \) sees each point of \( N \cap S \) via \( S \). Finally, set \( S \) is starshaped if and only if there is some point \( p \) in \( S \) such that \( p \) sees each point of \( S \) via \( S \), and the set of all such points \( p \) is called the (convex) kernel of \( S \), denoted \( \ker S \).

A theorem of Krasnosel'skii [6] states that if \( S \) is a nonempty compact set in \( \mathbb{R}^n \), then \( S \) is starshaped if and only if every \( n + 1 \) points of \( S \) are visible from a common point of \( S \). (A stronger result may be obtained by replacing points of \( S \) with boundary points of \( S \).) In [2], an analogue of the Krasnosel'skii theorem was proved for the dimension of this kernel: For each \( k \) and \( n \), \( 1 \leq k \leq n \), let 
\[
\begin{align*}
  f(n, n) &= n + 1 \\
  f(n, k) &= 2n 
\end{align*}
\]
if \( 1 \leq k \leq n - 1 \). If \( S \) is a compact set in \( \mathbb{R}^n \), then \( \ker S \) has dimension at least \( k \) if and only if for some \( \varepsilon > 0 \), every \( f(n, k) \) points of \( S \) see via \( S \) a common \( k \)-dimensional \( \varepsilon \)-neighborhood. Unfortunately, the uniform lower bound \( \varepsilon \) is necessary by an example in [3].

In order to obtain a theorem for the dimension of the kernel independent of this cumbersome \( \varepsilon \)-bound, we turn to the notion of clearly visible, previously appearing in work by Stavrakas [8] and Falconer [4]. In [1], analogues of the Krasnosel'skii theorem were obtained by replacing the concept of visible with that of clearly visible and by replacing points of \( S \) with lnc points of \( S \). A similar approach proves helpful here, and we have the following result: For \( f(n, k) \) defined above, \( 1 \leq k \leq n \), and for \( S \) a nonempty compact set in \( \mathbb{R}^n \), \( \ker S \) has dimension at least \( k \) if and only if every \( f(n, k) \) boundary points of \( S \) are clearly visible from a common \( k \)-dimensional
subset of S. If \( k = 1 \) or \( k = n \), the result is best possible. Moreover, if S is a compact, connected, nonconvex set in \( R^2 \), then the boundary of S may be replaced by the lnc points of S.

The following terminology will be used: \( \text{conv} \ S, \text{bdry} \ S, \) and \( \text{ker} \ S \) will denote the convex hull, boundary, and kernel, respectively, for set S. \( \text{Lnc} \ S \) will be the set of points of local nonconvexity of S, and if S is convex, \( \dim S \) will be the dimension of S. Finally, \( \sigma \) will represent the Hausdorff metric defined on the collection of compact, convex subsets of \( R^n \). The reader is referred to Valentine [9] for a discussion of these concepts and to Nadler [7] for further information on the Hausdorff metric.

2. The results. The first lemma is a variation of a result in [2].

**LEMMA 1.** For each \( k \) and \( n \), \( 1 \leq k \leq n \), let \( f(n, n) = n + 1 \) and \( f(n, k) = 2n \) if \( 1 < k < n - 1 \). Let \( \mathcal{B} \) be a uniformly bounded collection of compact convex sets in \( R^n \) which is closed with respect to the Hausdorff metric. Then for a \( k \) with \( 1 \leq k \leq n \), \( \dim \{ \bigcap \{ B : B \in \mathcal{B} \} \} \geq k \) if and only if every \( f(n, k) \) members of \( \mathcal{B} \) contain a common \( k \)-dimensional set.

**PROOF.** We need only establish the sufficiency of the condition. Observe that if every \( f(n, k) \) members of \( \mathcal{B} \) contain a common \( k \)-dimensional set, then by Helly's theorem in \( R^n \), \( \bigcap \{ B : B \in \mathcal{B} \} \neq \emptyset \). If \( k = n \), suppose on the contrary that \( \dim \{ \bigcap \{ B : B \in \mathcal{B} \} \} < n \). Then by a theorem of Falconer [4, Theorem 1], there exist \( r \) sets \( B_1, \ldots, B_r \) in \( \mathcal{B} \) such that \( \dim (B_1 \cap \cdots \cap B_r) = q \), where \( q < n \) and \( r \leq 2(n - q) \). However, by a result of Katchalski [5, Theorem a], every finite subfamily of \( \mathcal{B} \) has an \( n \)-dimensional intersection. We have a contradiction, our supposition is false, and the result is established when \( k = n \).

To establish the result for arbitrary \( k \), \( 1 \leq k \leq n \), we adapt an argument from [2, Lemma]. The inductive proof is sketched below. If \( n = 1 \), then \( k = 1 \), and we have our result. Assume the result is true for natural numbers less than \( n \), \( 2 \leq n \). If \( k = n \), again the proof is immediate, so assume \( 1 \leq k < n \). If \( \dim \{ \bigcap \{ B : B \in \mathcal{B} \} \} = n \), there is nothing to prove, so suppose that \( \dim \{ \bigcap \{ B : B \in \mathcal{B} \} \} < n \). Then by Falconer's result, there are \( r \) sets \( B_1, \ldots, B_r \) in \( \mathcal{B} \) with \( \dim (B_1 \cap \cdots \cap B_r) = q \), for some \( q < n \) and \( r \leq 2(n - q) \). Using Katchalski's theorem, \( q \geq k \).

Let \( G \) denote the \( q \)-dimensional flat in \( R^n \) determined by \( B_1 \cap \cdots \cap B_r \). Using our hypothesis and the fact that \( 1 \leq k \leq q < n \), it is not hard to show that every 2q sets in \( \mathcal{B} \) meet in a \( k \)-dimensional set in \( G \). Hence every \( f(q, k) \) members of \( \{ B \cap G : B \in \mathcal{B} \} \) meet in a \( k \)-dimensional set, and by our induction hypothesis, \( \dim \{ \bigcap \{ B \cap G : B \in \mathcal{B} \} \} \geq k \), finishing the induction and completing the proof of the lemma.

The following definitions will be helpful.

**DEFINITION.** Let \( S \) be a compact set in \( R^n \) and let \( q \in \text{bdry} \ S \). We define

\[
S_q = \{ x : [x, q] \subseteq S \},
\]

\[
A_q = \{ x : q \text{ is clearly visible from } x \text{ via } S \},
\]

and

\[
C_q = \bigcap \{ H : H \text{ a closed halfspace with } A_q \subseteq H \text{ and } q \in \text{bdry} \ H \}.
\]

Using these definitions, we establish two more lemmas.
Lemma 2. Let $S$ be a compact set in $\mathbb{R}^n$. If $B$ is in the \sigma-closure of $C \equiv \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$, then $B$ contains some set $A_q$.

Proof. Let $\{C_{q_n} \cap \text{conv } S : n \geq 1\}$ be a sequence in $C$ converging to $B$. Then since $S$ is bounded, some subsequence $\{q_{n_k}\}$ of $\{q_n\}$ converges to a point $q$, and clearly $q \in \text{bdry } S$. Since $\{C_{q_{n_k}} \cap \text{conv } S\}$ also converges to $B$, for convenience of notation we may assume that $\{q_n\}$ converges to $q$. We assert that $A_q \subseteq B$ for this particular $q$. Let $x \in A_q$. Then $q$ is clearly visible via $S$ from $x$, so for some neighborhood $N$ of $q$, $x$ sees via $S$ each point of $N \cap S$. There is some integer $M$ such that whenever $n > M$, $q_n \in N$. Hence for $n > M$, $q_n$ is clearly visible via $S$ from $x$, and so $x \in A_{q_n} \subseteq C_{q_n} \cap \text{conv } S$ for $n > M$. Since $\{C_{q_n} \cap \text{conv } S : n > M\}$ converges to $B$, $x \in B$. Thus $A_q \subseteq B$ and the lemma is established.

Corollary. Let $S$ be a compact set in $\mathbb{R}^n$, and let $k$ and $j$ be fixed, $0 < k < n$. If every $j$ sets in $\{A_q : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection, then every $j$ sets in the \sigma-closure of $C \equiv \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection.

Lemma 3. Let $S$ be a compact set in $\mathbb{R}^n$, and let $C = \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$. Then $\ker S = \bigcap\{B : B \in \text{the } \sigma\text{-closure of } C\}$.

Proof. For convenience, let $B$ denote the \sigma-closure of $C$. Using Lemma 2, clearly $\ker S \subseteq \bigcap\{A_q : q \in \text{bdry } S\} \subseteq \bigcap\{B : B \in B\}$, so we need to establish only the reverse inclusion. Let $x$ belong to $\mathbb{R}^n \sim \ker S$. By Krasnosel'skii's Lemma [9, Lemma 6.2], there is a $z$ in $\text{bdry } S$ and a closed halfspace $H$ such that $z \in \text{bdry } S$, $z \in H$, and $x \not\in H$. Since $A_z \subseteq S_z \subseteq H$, we have $A_z \subseteq H$, $x \not\in C_z$, and $x \not\in \bigcap\{B : B \in B\}$. Thus the lemma is proved.

Theorem 1. For each $k$ and $n$, $1 \leq k \leq n$, let $f(n, n) = n + 1$ and $f(n, k) = 2n$ if $1 \leq k \leq n - 1$. Let $S$ be a nonempty compact set in $\mathbb{R}^n$. Then for a $k$ with $1 \leq k \leq n$, $\dim \ker S \geq k$ if and only if every $f(n, k)$ points of $\text{bdry } S$ are clearly visible from a common $k$-dimensional subset of $S$. If $k = 1$ or $k = n$, the result is best possible.

Proof. The necessity of the condition is obvious. To establish its sufficiency, assume that every $f(n, k)$ points of $\text{bdry } S$ are clearly visible from a common $k$-dimensional subset of $S$. Using our hypothesis, every $f(n, k)$ members of $\{A_q : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection. Therefore, by the corollary to Lemma 2, every $f(n, k)$ sets in the \sigma-closure $B$ of $C \equiv \{C_q \cap \text{conv } S : q \in \text{bdry } S\}$ have at least a $k$-dimensional intersection. Standard arguments involving the Hausdorff metric [7, 9] show that $B$ is a uniformly bounded collection of compact convex sets in $\mathbb{R}^n$. Hence by Lemma 1, $\dim \bigcap\{B : B \in B\} \geq k$. Using Lemma 3, $\ker S = \bigcap\{B : B \in B\}$, so $\dim \ker S \geq k$, and the theorem is proved.

To see that the number $f(n, 1) = 2n$ is best possible, consider the following example, adapted from a construction by Katchalski [5, Theorem b, Case 1].

Example 1. For convenience of notation, use the component representation $(z_1, \ldots, z_n)$ for a point in $\mathbb{R}^n$, and let $D$ denote the closed unit ball centered at the origin. Construct a compact set $S$ in $\mathbb{R}^n$ such that $\text{Inc } S$ is a union of $2n$ disjoint convex sets $K_1, \ldots, K_{2n}$, each of dimension $n - 2$, and having the following properties:

1. For $x, y$ in $K_i, C_x = C_y$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(2) For $x$ in $K_i$, $1 \leq i \leq n$, $D \cap \{(z_1, \ldots, z_n) : z_i \geq 0\} \subseteq A_x \cap \{(z_1, \ldots, z_n) : z_i < 0\} = \emptyset$.

For $x$ in $K_{i+n}$, $1 \leq i \leq n$, $D \cap \{(z_1, \ldots, z_n) : z_i \leq 0\} \subseteq A_x \cap \{(z_1, \ldots, z_n) : z_i > 0\} = \emptyset$.

(3) $K_i \cap K_{i+n} \subseteq \{(z_1, \ldots, z_n) : z_i = 0\}, 1 \leq i \leq n$.

Then for $q_i \in K_i$, $1 \leq i \leq 2n$, the corresponding sets $A_{q_i}$ have only the origin in common. However, every $2n - 1$ of the sets in $\{A_q : q \in \text{bdry } S\}$ share a common interval in $D$. Thus the number $f(n, 1) = 2n$ is best possible.

Figure 1 below illustrates the example in $R^2$.

Similarly, our second example shows that the number $f(n, n) = n + 1$ is best.

**EXAMPLE 2.** Let $S$ be a compact set in $R^n$, with $\text{inc } S$ a union of $n + 1$ disjoint convex sets $K_1, \ldots, K_{n+1}$, each of dimension $n - 2$, and satisfying the following properties:

(1) For $x, y \in K_i$, $C_x = C_y$.

(2) Every $n$ members of $\{C_{q_i} : q_i \in K_i, 1 \leq i \leq n + 1\}$ meet in an $n$-dimensional set in $S$, yet $\bigcap\{C_{q_i} : q_i \in K_i, 1 \leq i \leq n\} = \emptyset$.

This may be done so that every $n$ members of $\{A_q : q \in \text{bdry } S\}$ meet in an $n$-dimensional set as well. Since $\bigcap\{A_q : q \in \text{bdry } S\} = \emptyset$, the number $f(n, 1) = n + 1$ is best possible.

We close with a stronger version of Theorem 1 which holds in the plane.

**THEOREM 2.** Let $S$ be a compact, connected, nonconvex set in $R^2$. Then for $k = 1$ or $k = 2$, $\dim \ker S \geq k$ if and only if every $g(k) = \max\{3, 6 - 2k\}$ inc points of $S$ are clearly visible from a common $k$-dimensional subset of $S$. The result is best possible.

**PROOF.** By [1, Lemma 4], if $S$ is a closed connected set in $R^2$, then $\ker S = \bigcap\{C_q \cap \text{conv } S : q \in \text{inc } S\}$. Using this result, it is not hard to show that Lemmas 2 and 3 above hold when $S$ is a compact, connected, nonconvex set in $R^2$ and when $\text{bdry } S$ is replaced by $\text{inc } S$. An easy adaptation of the proof of Theorem 1 completes the argument.

Examples 1 and 2 of [1] show that the result is best possible.
References


Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73019