A CELLULAR MAP BETWEEN NONHOMEOMORPHIC POLYHEDRA
WHOSE NONDEGENERACY SET IS A NULL SEQUENCE
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ABSTRACT. A cellular map between nonhomeomorphic polyhedra whose
nondegeneracy set consists of a null sequence of cellular sets is constructed.
The construction begins with Daverman’s example of a cellular map between
nonhomeomorphic polyhedra and uses an amalgamation technique.

Early results obtained concerning cellular maps between polyhedra [6, 7] seemed
to indicate that such maps might indeed behave as nicely as cellular maps between
n-manifolds, n ≠ 4, namely, such maps are approximable by homeomorphisms
[1, 5, 10]. However, Daverman recently constructed an example of a cellular map
between nonhomeomorphic polyhedra. (See [8].) The knowledge that cellular maps
between polyhedra need not always be approximable by homeomorphisms leads one
to try to find a “simplest” such example. Since any cellular map having only a finite
number of nondegenerate point preimages is approximable by homeomorphisms, the
next case to consider would be a map with a countable number of nondegenerate
point preimages, each being a cellular subset of the domain. In light of the Bing
shrinking criterion, one might be led to expect that if this sequence of cellular
sets was in fact a null sequence, then the cellular map would be approximable by
homeomorphisms. However, examples of Bing [3], Daverman [4], and Wright [12]
of nonshrinkable decompositions of manifolds with a null sequence of cellular sets
as the nondegeneracy set indicate that more is involved. In fact, one can modify
Daverman’s cellular map between nonhomeomorphic polyhedra to obtain a cellular
map between polyhedra, with nondegeneracy set consisting of a null sequence of
cellular sets, which cannot be approximated by homeomorphisms. The purpose of
this paper is to construct that map.

1. Preliminaries. A polyhedron P is a subset of some Euclidean space R^n
such that each point b ∈ P has a neighborhood N = bl, the join of b and a
compact subset L of P [9]. A homotopy h_t: P → P, 0 ≤ t ≤ 1, for which h_t is a
homeomorphism when t < 1 is a pseudoisotopy. A compact subset X of P is cellular
in P if there is a pseudoisotopy h_t: P → P such that X is the only nondegenerate
point preimage of h_1. A proper surjection f: P → Q between polyhedra is a cellular
map if for each y ∈ Q, f⁻¹(y) is a cellular subset of P. The nondegeneracy set of
f is N_f = {f⁻¹(y)| y ∈ Q and f⁻¹(y) is not a point}.
The intrinsic dimension of a point x in P, denoted I(x, P), is given by I(x, P) =
max{n ∈ Z| there is an open embedding h: R^n × cL → P with L a compact
polyhedron and h(R^n × cL) a neighborhood of h(0 × c) = x}, where cL is the open
cone on $L$ [2]. A cellular neighborhood in $P$ is an open subset of $P$ homeomorphic to $\mathbb{R}^n \times cL$, again with $L$ a compact polyhedron and $cL$ the open cone on $L$. The intrinsic $n$-skeleton of $P$ is $P^{(n)} = \{ x \in P | I(x, P) \leq n \}$, and the intrinsic $n$-stratum of $P$ is $P[n] = P^{(n)} - P^{(n-1)}$. Note that $P[n]$ is a topological $n$-manifold.

One theorem from [6] will be stated here for later reference.

**Theorem 1.** The following are equivalent:

1. $X$ is cellular in $P$.
2. The projection $\pi: P \to P/X$ is approximable by homeomorphisms.
3. $X = \bigcap_{i=1}^{\infty} W_i$, where the $W_i$'s are homeomorphic cellular neighborhoods with $W_{i+1} \subseteq W_i$.

**2. Daverman's example.** Let $W$ be a contractible $(n+1)$-manifold, $n \geq 3$, whose boundary is a nonisomorphically connected homology $n$-sphere $H$, and choose $w \in \text{int} W$. The polyhedron $P$ is given by

$$P = (W \times S^1) \cup_{w \times S^1} (W \times S^1),$$

two copies of $W \times S^1$ identified along $w \times S^1$ by the identity map. Let $\tilde{W}$ be a submanifold of $W$ such that $\tilde{W}$ is homeomorphic to $W$, $w \in \tilde{W}$, and $W - \text{int} \tilde{W} \simeq H \times [0,1]$.

The polyhedron $Q$ is defined by

$$Q = (cH \times S^1) \cup_{c \times S^1} (cH \times S^1),$$

with $cH$ being the standard cone on the homology sphere $H$. The cellular map $f: P \to Q$ takes each $(W \times y) \cup_{w \times y} (W \times y)$ to $(cH \times y) \cup (cH \times y)$ by sending the cellular subset $(\tilde{W} \times y) \cup_{w \times y} (\tilde{W} \times y)$ to the point $c \times y$. (See [7] for a more detailed discussion of the example.)

**3. Modifying cellular maps.** In order to construct the desired example, it will be necessary to modify the map $f: P \to Q$ by shrinking out certain nondegenerate point preimages. The following proposition and Theorem 1 are needed to conclude that the resulting map is cellular. The statement of the proposition is a slight variation on that of Proposition 1.3 of Walsh [11], and the proof included here is essentially his.

**Proposition 2.** Let $\tilde{f}: \tilde{P} \to \tilde{Q}$ be a cellular map between compact polyhedra. If $\tilde{h}_t: \tilde{P} \to \tilde{P}$, $0 \leq t < 1$, is an isotopy such that $\tilde{g} = \lim_{t \to 1} \tilde{f} \circ \tilde{h}_t: \tilde{P} \to \tilde{Q}$ is a continuous function, then $\tilde{g}$ is also cellular.

**Proof.** Choose $y \in \tilde{Q}$. Since $\tilde{f}^{-1}(y)$ is cellular in $\tilde{P}$, it follows from Theorem 1 that there is a sequence of homeomorphic cellular neighborhoods $\{W_i\}_{i \in \mathbb{N}}$ such that $W_{i+1} \subseteq W_i$ and $\tilde{f}^{-1}(y) = \bigcap_{i=1}^{\infty} W_i$. Given $\epsilon > 0$, it is sufficient to show that there is some $t$, $0 \leq t < 1$, and integer $k$ such that

$$\tilde{g}^{-1}(y) \subseteq \tilde{h}_t^{-1}(W_k) \subseteq N_\epsilon(g^{-1}(y)).$$

Let $U \subseteq \tilde{g}(N_\epsilon(\tilde{g}^{-1}(y)))$ be an open neighborhood of $y$ with $\tilde{g}^{-1}(U) \subseteq N_\epsilon(g^{-1}(y))$. There is a $t' < 1$ such that for $t \geq t'$, $(\tilde{f} \circ \tilde{h}_t)(y) \subseteq \tilde{g}^{-1}(U)$. Choose $k$ so that $\tilde{f}(W_k) \subseteq U$ and $(\tilde{f} \circ \tilde{h}_t)^{-1}(\tilde{f}(W_k)) \subseteq \tilde{g}^{-1}(U)$ for $t \geq t'$. Let $U' \subseteq \tilde{f}(W_k)$ be an
open neighborhood of $y$ with $\tilde{f}^{-1}(U') \subseteq W_k$. There is an $s$, $1 > s > t'$, such that for $t > s$, $\tilde{g}^{-1}(y) \subseteq (f \circ \tilde{h}_t)^{-1}(U')$. Now for $1 > t > s$,

$$\tilde{g}^{-1}(y) \subseteq (\tilde{f} \circ \tilde{h}_t)^{-1}(U') = \tilde{h}_t^{-1}(\tilde{f}^{-1}(U')) \subseteq \tilde{h}_t^{-1}(W_k)$$

$$\subseteq \tilde{h}_t^{-1}(\tilde{f}^{-1}(\tilde{f}(W_k))) = (f \circ \tilde{h}_t)^{-1}(f(W_k)) \subseteq \tilde{g}^{-1}(U) \subseteq N_{\varepsilon}(\tilde{g}^{-1}(y)).$$

Thus we have $\tilde{g}^{-1}(y) \subseteq \tilde{h}_t^{-1}(W_k) \subseteq N_{\varepsilon}(g^{-1}(y))$ for some $t < 1$.

The example. Let $f: P \to Q$ be Daverman's example described earlier, and let $D = \{y_i\}$ be a countable dense subset of $c \times S^1 = Q[1]$. Since each $f^{-1}(y_i)$ is cellular in $P$, there exist cellular neighborhoods $W_{i,k}$ such that $f^{-1}(y_i) = \bigcap_{k=1}^{\infty} W_{i,k}$. Choose $k(1)$ so that $\text{diam}(W_{1,k(1)}) < 1$. It follows from Proposition 1.5 of [6] that there is a homeomorphism $\tilde{H}_1: P \to P$ such that

1. $\tilde{H}_1$ is isotopic to the identity map on $P$ with compact support in $W_{1,k(1)} - P[1]$.

2. $\text{diam} \tilde{H}_1(f^{-1}(y_1)) < 1$.

Inductively now, choose an integer $k(i) > k(i - 1)$ such that $f(W_{r,k(i)}) \cap f(W_{s,k(i)}) = \emptyset$ for $1 \leq r, s \leq i$ and $r \neq s$, and $\text{diam} f(W_{r,k(i)}) < 1/2^i$. As above, there is a homeomorphism $H_i: P \to P$ such that

1. $H_i$ is isotopic to $H_{i-1}$ with compact support in $H_{i-1}(\bigcup_{r \leq i} W_{r,k(i)}) - P[1]$.

2. $\text{diam} H_i(f^{-1}(y_i)) < 1/i$ for $1 \leq r \leq i$.

Now let $h_i: P \to P$ be the isotopy obtained by piecing together the above isotopies. That is, by defining $h_i / (i + 1) = H_i$, and letting $h_i: P \to P$, $i / (i + 1) \leq t \leq (i + 1) / (i + 2)$, be the isotopy between $H_i$ and $H_{i+1}$. Then $\lim_{t \to 1} f \circ h_t^{-1} = \tilde{f}$ will be a continuous function from $P$ onto $Q$ which is cellular by Proposition 2. Note that $\tilde{f}$ is 1-1 over $D$. Also, we apply Theorem 4.2 of [6] to approximate the restriction of $\tilde{f}$ to $P - f^{-1}(Q[1])$ by a homeomorphism which extends to $\tilde{f}$ on $\tilde{f}^{-1}(Q[1])$. Thus it may be assumed that $\tilde{f}$ is 1-1 over $(Q - Q[1]) \cup D$.

At this point, the image of the nondegeneracy set of $\tilde{f}$ is 0-dimensional. The amalgamation idea used by Edwards [5], Daverman [4], and Wright [12] is now employed. Let $N_i = \{f^{-1}(y) \cap f^{-1}(y) : \text{diam} f^{-1}(y) \geq 1/i\}$. Each $N_i$ is a closed subset of $P$, and hence $\tilde{f}(N_i)$ is a closed subset of $Q[1]$. Cover $\tilde{f}(N_i)$ with a finite collection of disjoint arcs $A_1, \ldots, A_{k(i)}$, each having endpoints in $D$. Inductively, $\tilde{f}(N_{i+1}) - \tilde{f}(N_i)$ is a closed subset of $Q[1] - \bigcup_{j < k(i)} A_j$. Cover $\tilde{f}(N_{i+1}) - \tilde{f}(N_i)$ with a finite collection of arcs $A_{k(i) + 1}, \ldots, A_{k(i+1)}$, each having diameter less than $1/(i + 1)$ and lying in $Q[1] - \bigcup_{j \leq k(i)} A_j$.

Consider the collection $\{A_i\}_{i \in \mathbb{N}}$ of arcs in $Q[1]$. It is a null collection of arcs, and by the choice of the $N_i$'s, $\{f^{-1}(A_i)\}_{i \in \mathbb{N}}$ will also be a null sequence. It remains to show that each $\tilde{f}^{-1}(A_i)$ is cellular in $P$ and that the decomposition $G$ of $P$ whose nondegenerate elements are $\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbb{N}}$ yields $Q$. Then $\pi: P \to \pi(P/\tilde{f}^{-1}(A_i))_{i \in \mathbb{N}} \simeq Q$ will be the desired map.

First note that $Q/\{A_i\}_{i \in \mathbb{N}} \simeq Q$. Identifying $c \times S^1$ with $S^1$, there is a pseudoisotopy $g_1: S^1 \to S^1$ such that the degenerate point preimages of $g_1$ are $\{A_i\}_{i \in \mathbb{N}}$. Letting $cH^n = (H^n \times [0,2])/(H^n \times \{0\})$, define $G_r: cH^n \times S^1 \to cH^n \times S^1$ by

$$G_r(x, t, s) = \begin{cases} (x, t, g_t(1-t)(s)), & 0 \leq t \leq 1, \\ (x, t, s), & t \geq 1, \end{cases}$$
where \( r \in [0,1] \), \( x \in H^n \), \( t \in [0,2] \), and \( s \in S^1 \). Now \( G_1: Q \to Q \) has \( \{A_i\}_{i \in \mathbb{N}} \) as its nondegeneracy set, and it follows that \( Q/\{A_i\}_{i \in \mathbb{N}} \simeq Q \).

Also, the composition \( G_1 \circ f: P \to Q \) is a cellular map where nondegenerate point preimages are \( \{B_i \times \tilde{W}\}_{i \in \mathbb{N}} \), where \( B_i \) is the arc in \( P[1] \) which is mapped homeomorphically onto \( A_i \) by \( f \). If we consider the map \( f^*: \lim_{t \to 1} G_1 \circ f \circ h_t^{-1} = G_1 \circ \tilde{f} \), then again by Proposition 2, \( f^* \) is a cellular map. But if \( y_i = G_1(A_i) \), then

\[
(f^*)^{-1}(y_i) = \lim_{t \to 1} h_t(f^{-1} \circ G_1^{-1}(y_i)) = \lim_{t \to 1} h_t(f^{-1}(A_i)) = \tilde{f}^{-1}(A_i).
\]

Thus each \( \tilde{f}^{-1}(A_i) \) is cellular in \( P \). Now \( \pi: P \to P/\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbb{N}} \) is a cellular map and \( P/\{\tilde{f}^{-1}(A_i)\}_{i \in \mathbb{N}} \simeq \tilde{f}(P)/\{A_i\}_{i \in \mathbb{N}} \simeq Q/\{A_i\}_{i \in \mathbb{N}} \simeq Q \). Thus \( \pi: P \to Q \) is the desired map.

REFERENCES

12. D. Wright, *A decomposition of \( E^n \) (n \( \geq \) 3) into points and a null sequence of cellular sets which yields a non-manifold decomposition space*, General Topology Appl. 10 (1979), 297–304.