ABSTRACT. An operation on the set of partitions of a number \( n \) is introduced and the possible cycles are determined.

The problem to be discussed in the following has been circulating for some time. Let \( S \) be a natural number and let \( \lambda \) be a partition of \( S \) of length \( l \) with parts \( \lambda_1, \lambda_2, \ldots, \lambda_l \). Define \( T(\lambda) \) as the partition of \( S \) with parts \( \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_l - 1, l \), ignoring any zeros that might occur. Assume that \( S = 1 + 2 + \cdots + N \). Then it turns out that repeated application of \( T \) always yields the partition \( 1, 2, \ldots, N \). We shall prove that this is indeed so.

If \( S \) is arbitrary, repeated application of \( T \) leads into a cycle of partitions since there are only a finite number of these. We want to determine all cycles corresponding to \( S \). Now, a cycle of partitions is completely determined by the sequence of the consecutive lengths of the partitions in the cycle. Due to this observation we define sets \( M_n \) by

\[
M_n = \{ \sigma = (\sigma_i)_{i \in \mathbb{Z}} \mid \max \sigma_i = n, \forall i: \sigma_i = |\{ \sigma_j | j < i, \sigma_j \geq i - j \}| \}.
\]

A cycle of partitions then corresponds to a \( \sigma \in M_n \), where \( n \) is the maximal length of a partition in the cycle. If we define \( \sigma, \tau \in M_n \) to be equivalent if there exists an integer \( c \) such that \( \sigma_{i+c} = \tau_i \) for all \( i \), then the cycle determines a unique class in \( M_n \). Conversely, a class in \( M_n \) uniquely determines a cycle of partitions by regarding the \( \sigma_i \)'s as the lengths of the partitions in a cycle.

The above discussion shows that we may find all cycles of partitions by determining the set of equivalence classes in \( M_n \) and this is what we shall do. First, it is not hard to see that if \( \sigma \in M_n \) then there exists a \( p \in \mathbb{N} \) such that \( \sigma_{i+p} = \sigma_i \) for all \( i \). We call the smallest such \( p \) the period of \( \sigma \) and denote it by \( p(\sigma) \).

**Lemma 1.** Let \( \sigma \in M_n \). If \( \sigma_i = n \) then \( \forall k \in \mathbb{Z}: \sigma_{i+kn} = n \).

**Proof.** By definition of \( M_n \), \( \sigma_{i-n} = n \) since \( n = \max \sigma_i \). Hence the lemma holds for negative \( k \). Now \( \sigma_{i+n} = n \), so by what we just observed \( \sigma_{i+n} = n \) as well and we are done. Q.E.D.

**Proposition 2.** Let \( \sigma \in M_n \). Then \( \forall i: \sigma_i \in \{n, n-1\} \).

**Proof.** We may assume \( n > 1 \). Define \( \sigma' = (\sigma'_i)_{i \in \mathbb{Z}} \) by

\[
\begin{align*}
\sigma'_i &= \sigma_i & \text{if } \sigma_i < n, \\
\sigma'_i &= n - 1 & \text{if } \sigma_i = n.
\end{align*}
\]

Using Lemma 1 it follows that \( \sigma' \in M_{n-1} \). We may assume \( \sigma_0 = n \). Now, for all
i \in \mathbb{Z} \text{ there exist integers } x \text{ and } y \text{ such that } i = xn + y(n - 1). \text{ Then by Lemma 1 we obtain }$
\sigma'_{xn} = n - 1 \Rightarrow \sigma'_i = \sigma'_{xn + y(n - 1)} = n - 1.$
Hence \( \sigma' = (n - 1)i \in \mathbb{Z} \) which implies the result about \( \sigma \). Q.E.D.

**Corollary 3.** \( \sigma \in M_n \Rightarrow p(\sigma)|n \).

**Proof.** Combine Lemma 1 with Proposition 2. Q.E.D.

**Theorem 4.** Let \( \sigma \in M_n \) and assume \( S_\sigma = 1 + 2 + \cdots + N \). Then \( N = n \) and \( M_n \) has just one class with sum \( S_\sigma \), namely \( \sigma = (n)_{\in \mathbb{Z}} \).

**Proof.** Let \( \sigma_1, \sigma_2, \ldots, \sigma_n \) consist of \( n - 1 \)'s and \( n - a \) n's. Then \( S_\sigma = \binom{n+1}{2} - a, 0 \leq a \leq n - 1 \). Simple arithmetic now shows that \( n = N \) and \( a = 0 \). Q.E.D.

Theorem 4 proves that if we start with any partition of \( S = 1 + 2 + \cdots + N \) and apply \( T \) a number of times then we always end with the partition \( 1, 2, \ldots, N \) (corresponding to \( \sigma = (n)_{\in \mathbb{Z}} \)).

To determine all cycles for arbitrary \( S \) notice that there exist unique \( n \) and \( a \) such that \( S = \binom{n+1}{2} - a, 0 \leq a \leq n - 1 \). Let \( C_a(n) \) denote the number of classes in \( M_n \) for which \( \sigma_1, \ldots, \sigma_n \) contain \( n - 1 \)'s and \( n - a \) n's. Then \( C_a(n) \) is the number of cycles of partitions of \( S \).

**Theorem 5.** \( C_a(n) = \frac{1}{n} \sum_{d|(n,a)} \varphi(d) \frac{n/d}{(n/a)!} \), where \( \varphi \) is Euler's function.

**Proof.** If \( \sigma_1, \ldots, \sigma_n \) is any sequence consisting of \( n - 1 \)'s and \( n - a \) n's, we define the sequence \( \sigma \) by periodically extending to both sides. It is clear that \( \sigma \in M_n \). Thus \( C_a(n) \) is equal to the number of circular words of length \( n \) with \( a \) letters \( n - 1 \), and \( n - a \) letters \( n \). The standard Pólya enumeration theory (see e.g. [1]) applies. If one carries out the details one obtains the formula in the theorem. Q.E.D.

As an example of how to find all cycles corresponding to a specific \( S \) we consider \( S = 8 \). Here \( n = 4, a = 2 \), so there are \( C_2(4) = 2 \) cycles. The classes in \( M_4 \) corresponding to these cycles are represented by \( \ldots, 4, 3, 4, 3, 4, 3, \ldots \) and \( \ldots, 4, 4, 3, 3, 4, 4, \ldots \). Transforming this into cycles we get

\[
\begin{array}{c}
1 & 1 & 2 & 4 \\
\downarrow & & \downarrow & \\
2 & 2 & 4 & \\
\uparrow & \downarrow & \uparrow & \downarrow \\
1 & 1 & 3 & 3 \\
\end{array}
\]

Finally we have some further comments on the special case \( S = 1 + 2 + \cdots + N \). Theorem 4 shows that the partitions of \( S \) can be arranged in a tree so that the vertices correspond to the partitions and going down corresponds to applying \( T \). The root of the tree is of course the partition \( 1, 2, \ldots, N \). For \( S = 3, 6 \) and 10 one may draw these trees by hand but this soon becomes impracticable since the number of partitions grows rapidly. For \( S = 15 \) there are 176 partitions. A
Cycles of Partitions

Figure 1
computer program was written in collaboration with C. B. Hansen to generate and
draw these trees. We have included the tree for $S = 15$ as Figure 1. The vertices
are labeled with the length of the partitions. Notice that there are $21 = 5^2 - 5 - 1$
levels in the tree. This appears to generalize to $N^2 - N + 1$ levels in the tree for
$S = 1 + 2 + \cdots + N$.

REFERENCES


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