PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 85, Number 4, August 1982

SCHUR INDICES IN FINITE QUATERNION-FREE GROUPS

A. D. OH

ABSTRACT. Let G be a finite, quaternion-free group with exponent e, let F be a field of characteristic zero and let χ be an absolutely irreducible character of G. Suppose that a Sylow 2-subgroup of the Galois group of $F(\sqrt[e]{1})$ over F is cyclic. It is shown that if χ is not real valued, then the Schur index of χ over F is odd.

Let G be a finite group with exponent e and let F be a field of characteristic zero. We denote the set of absolutely irreducible characters of G by $\mathrm{Irr}(G)$, and write $m_F(\chi)$ for the Schur index over F of $\chi \subset \mathrm{Irr}(G)$. ξ_n denotes a primitive nth root of unity. We say that $\chi \in \mathrm{Irr}(G)$ is real valued if the field $Q(\chi)$ is embedded in R, the field of real numbers. Following [**6**] we call G a quaternion-free group if a generalized quaternion group is not involved in G.

In [**3**], B. Fein proved that if -1 is a sum of two squares in F and a Sylow 2-subgroup of the Galois group $\mathrm{Gal}(F(\xi_e)/F)$ is cyclic, then $m_F(\chi)$ is odd. Because of the well-known Brauer-Speiser theorem, we investigate $m_F(\chi)$ when χ is not real valued. We prove the following.

THEOREM. *Let G be a finite, quaternion-free group with exponent e, let F be a field of characteristic zero and let $\chi \in \mathrm{Irr}(G)$. Suppose that a Sylow 2-subgroup of the Galois group $\mathrm{Gal}(F(\xi_e)/F)$ is cyclic. If χ is not real valued, then $m_F(\chi)$ is odd.*

The assumption that G is quaternion-free in the theorem cannot be dropped as the following example shows. Let G be a nilpotent group of order $8q$ where q is a prime such that $q \equiv 7 \pmod 8$, and assume that its Sylow 2-subgroup is generalized quaternion. Choose a faithful, complex irreducible character χ of G and let $F = Q(\chi)$. Clearly, $F = Q(\xi_q)$ and $F(\xi_e) = F(\sqrt{-1})$. To see that $m_F(\chi) = 2$, we consider an F-triple (G, X, χ) where $X \leqslant G$ with $|G : X| = 2$. Since $q \equiv 7 \pmod 8$, -1 is not a norm in $F(\sqrt{-1})/F$ by [**1**, Note, p. 173], thus $m_F(\chi) = 2$ by Theorem 2.2 of [**2**].

The proof of the theorem is based on reductions of G and χ to an F-triple as in [**2**], and norm computations in reduced field extensions utilizing Theorem 2.2 of [**2**].

LEMMA 1. *If (H, X, θ) is an F-triple and K is a subfield of $F(\theta)$, then (H, X, θ) is a K-triple.*

PROOF. If K is a subfield of F, then (H, X, θ) is a K-triple by Lemma 9.17 (b) of [**1**]. Since (H, X, θ) is an $F(\theta)$-triple, the result follows.

Received by the editors December 27, 1981.
1980 *Mathematics Subject Classification*. Primary 20C15.

LEMMA 2. *Let* (H, X, θ) *be an F-triple. Suppose that* H/X *is a cyclic 2-group and* $|X \cap Y| = 2$ *for a cyclic supplement* Y *to* X *in* H. *Then there is an F-triple* (YX_0, X_0, θ_0) *with* $X_0 \leqslant X$ *and* $F(\theta_0) \subseteq F(\theta)$, *such that* (i) X_0 *is a 2-group or* $|X_0| = 2q$ *for an odd prime* q, (ii) $Y \cap X = Y \cap X_0$, *and* (iii) $m_F(\theta) = 1$ *if* $m_F(\theta_0) = 1$.

PROOF. Let λ be an irreducible constituent of θ_X, and write $\pi(X)$ for the set of primes dividing $|X|$. By Lemma 2.1 of **[2]**, $\mathrm{Gal}(F(\lambda)/F(\theta))$ is a cyclic 2-group, thus $\{F(\theta)(\xi_{q^i})\colon q \in \pi(X) \text{ and } q^i \| |X|\}$ is linearly ordered; moreover, $F(\theta)(\xi_{q^i}) = F(\theta)(\xi_q)$ if $q \in \pi(X)$ is odd and $i > 1$. Thus we have either (a) or (b) below;

(a) $F(\lambda) = F(\theta)(\xi_q)$ for some odd prime $q \in \pi(X)$.

(b) $F(\lambda) = F(\theta)(\xi_s)$ where s is the order of a Sylow 2-subgroup of X.

In case (a), define $X_0 = A(X \cap Y)$ where $A \leqslant X$ with $|A| = q$, so that $|X_0| = 2q$ and $Y \cap X = Y \cap X_0$; in case (b), define X_0 to be a Sylow 2-subgroup of X. Letting $\theta_0 = \theta_{YX_0}$, we see that (YX_0, X_0, θ_0) is an $F(\theta)$-triple; hence it is an F-triple by Lemma 1. Since $F(\lambda) = F(\theta)(\lambda_{X_0})$, Theorem 2.2 of **[2]** implies that $m_F(\theta) = 1$ if and only if $m_{F(\theta)}(\theta_0) = 1$. Thus, $m_F(\theta) = 1$ if $m_F(\theta_0) = 1$.

PROOF OF THEOREM. By Goldschmidt-Isaacs Theorem **[2**, Theorem 1] and $m_F(\chi) = m_{F(\chi)}(\chi)$, we may assume $\sqrt{-1} \notin F$ and $F = F(\chi)$. Suppose that $2 \mid m_F(\chi)$. Then, by Theorem 2 of **[2]**, there exists an F-triple (H, X, θ) such that H is a section of G, H/X is a 2-group, $2 \mid m_F(\theta)$ and $2 \nmid |F(\theta)\colon F|$. Note that H/X is cyclic since H/X is a section of $\mathrm{Gal}(F(\xi_e)/F)$ by Lemma 2.1 of **[2]**, and since $\mathrm{Gal}(F(\xi_e)/F)$ has a cyclic Sylow 2-subgroup.

Let Y be a cyclic supplement to X in H such that $X \cap Y$ is a 2-group. Since $\theta \in \mathrm{Irr}(H)$ is faithful and $X \cap Y$ is central in H, $F(\theta)$ contains a primitive $|X \cap Y|$th root of unity and thus $|X \cap Y| \leqslant 2$. The case $|X \cap Y| = 1$ does not occur by Theorem 2.2 of **[2]**. Hence $|X \cap Y| = 2$. Now, by Lemma 2 we reduce (H, X, θ) to the case where X is a 2-group or $|X| = 2q$ for an odd prime q.

If X is a 2-group, then Theorem 2.13 of **[6]** implies that a generalized quaternion group is involved in H (hence in G), against our assumption on G.

Next, suppose $|X| = 2q$ and assume $|Y| = 4$. Put $F_1 = F \cap Q(\xi_e)$ and let λ be an irreducible constituent of θ_X. Then (H, X, θ) is an F_1-triple (by Lemma 1) and $2 \mid m_{F_1}(\theta)$; it follows from Theorem 2.2 of **[2]** that -1 cannot be a norm in $F_1(\lambda)/F_1(\theta)$. On the other hand, we observe that $|F_1(\lambda)\colon F_1(\theta)| = 2$ (by Lemma 2.1 of **[2]**), $\sqrt{-1} \notin F_1(\theta)$ (since $|F(\theta)\colon F|$ is odd) and that $F_1(\theta)(\sqrt{-1}) = F_1(\theta)(\xi_q) = F_1(\lambda)$ (since $\mathrm{Gal}(Q(\xi_e)/F_1)$ has a cyclic Sylow 2-subgroup). Put $K = F_1(\theta)$ and $L = F_1(\lambda)$. We compute the norm residue symbol $(-1, L/K)_{\mathfrak{p}}$ where $\mathfrak{p}$ is a K-prime. Since χ is not real valued, $F_1 \not\subseteq R$ and thus $(-1, L/K)_{\mathfrak{p}} = 1$ if $\mathfrak{p}$ is infinite; since -1 is a unit, $(-1, L/K)_{\mathfrak{p}} = 1$ if $\mathfrak{p}$ is a finite K-prime unramified in L/K. If $\mathfrak{p}$ is a finite K-prime ramified in L/K then Corollary 6-2-4 of **[4]** implies that $\mathfrak{p}$ divides q since $L = K(\xi_q)$, and that $\mathfrak{p}$ divides 2 since $L = K(\sqrt{-1})$; this is impossible. Hence $(-1, L/K)_{\mathfrak{p}} = 1$ for all K-primes $\mathfrak{p}$, and the norm theorem **[5**, p. 392] implies that -1 is a norm in L/K, a contradiction.

For the remaining case where $|X| = 2q$ and $|Y| > 4$, the following Lemma yields a contradiction, which will complete the proof of Theorem.

Lemma. *Let (H, X, θ) be an F-triple such that $H = YX$ for a cyclic 2-group Y with $|X \cap Y| = 2$ and that $|X| = 2q$ for an odd prime q. Suppose that* $\mathrm{Gal}(F(\xi_n)/F)$ *has a cyclic Sylow 2-subgroup, where n is the exponent of H. If $|Y| > 4$, then $m_F(\theta) = 1$.*

Proof. Let λ be an irreducible constituent of θ_X. We will show that -1 is a norm in $F_0(\lambda)/F_0(\theta)$ for a subfield F_0 of F; it follows then by Theorem 2.2 of **[2]** and Lemma 1 that $m_{F_0}(\theta) = 1$, and hence $m_F(\theta) = 1$.

Let $|H| = 2^t q$. By Lemma 2.1 of **[2]**, $|F(\lambda) : F(\theta)| = 2^{t-1}$. Since $\mathrm{Gal}(F(\xi_n)/F)$ has a cyclic Sylow 2-subgroup and $|F(\theta)(\xi_{2^t}) : F(\theta)| \leq 2^{t-1}$, it follows that $\xi_{2^t} \in F(\lambda)$ and hence $F(\lambda) = F(\xi_q) = F(\xi_n)$. If $q \equiv 3 \pmod 4$, Lemma 2.1 of **[2]** would yield $t < 3$, which is not the case. Thus $q \equiv 1 \pmod 4$. Set $F_0 = F \cap Q(\xi_n)$, and write $K = F_0(\lambda)$, $k = F_0(\theta)$; so the extension K/k is cyclic of degree 2^{t-1} by Lemma 2.1 of **[2]** and Lemma 1.

Now, we show that -1 is a norm in K/k, i.e., that the norm residue symbol $(-1, K/k)_{\mathfrak{p}} = 1$ for all k-primes $\mathfrak{p}$. It follows from $F(\lambda) = F(\xi_q) = F(\xi_n)$ that

$$\mathrm{Gal}(Q(\xi_n)/F_0) \cong \mathrm{Gal}(F(\lambda)/F) \cong \mathrm{Gal}\big(Q(\xi_q)/F \cap Q(\xi_q)\big),$$

and $K = k(\xi_q) = k(\xi_n)$. Note that 4 divides $|F(\lambda) : F|$, and $\mathrm{Gal}(Q(\xi_n)/F_0)$ is cyclic; but, the structure of $\mathrm{Gal}(Q(\xi_n)/Q)$ shows that if E is a subfield of $Q(\xi_n) \cap R$ with $|Q(\xi_n) : E| = 4$, then $\mathrm{Gal}(Q(\xi_n)/E)$ is a four-group. Hence $F_0 \not\subseteq R$; and thus $(-1, K/k)_{\mathfrak{p}} = 1$ for infinite primes $\mathfrak{p}$ of k. Thus by Corollary 6-2-4 of **[4]**, we need only show that $(-1, K/k)_{\mathfrak{p}} = 1$ for finite k-primes $\mathfrak{p}$ dividing q. Since $q \equiv 1 \pmod 4$, $\sqrt{-1} \in Q_q$. By Goldschmidt-Isaacs Theorem [**2**, Theorem 1], we have $m_{k_{\mathfrak{p}}}(\theta) = 1$ and thus $(-1, K/k)_{\mathfrak{p}} = 1$. This completes the proof of Lemma.

References

1. I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York, 1976.
2. D. M. Goldschmidt and I. M. Isaacs, *Schur indices in finite groups*, J. Algebra **33** (1975), 191–199.
3. B. Fein, *Schur indices and sums of squares*, Proc. Amer. Math. Soc. **51** (1975), 31–34.
4. L. J. Goldstein, *Analytic number theory*, Prentice-Hall, Englewood Cliffs, N. J., 1971.
5. S. Iyanaga, *The theory of numbers*, North-Holland, Amsterdam, 1975.
6. L. Dornhoff, *M-groups and 2-groups*, Math. Z. **100** (1967), 226–256.

Department of Mathematics, Stillman College, Tuscaloosa, Alabama 35401