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A NOTE ON PREPROJECTIVE PARTITIONS OVER HEREDITARY ARTIN ALGEBRAS

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ABSTRACT. If Λ is an artin algebra there is a partition of ind Λ, the category of indecomposable finitely generated Λ-modules, ind $\Lambda = \bigcup_{i \geq 0} \underline{\underline{P}}_i$, called the preprojective partition. We show that $\underline{\underline{P}}_i$ can be easily constructed for hereditary artin algebras, if $\underline{\underline{P}}_{i-1}$ is known: A is in $\underline{\underline{P}}_i$ if and only if A is not in $\underline{\underline{P}}_{i-1}$ and there is an irreducible map $B \to A$, where B is in $\underline{\underline{P}}_{i-1}$.

If Λ is a hereditary artin algebra V. Dlab and C. M. Ringel called an indecomposable module X preprojective if there exist an indecomposable projective P and an integer j such that $X \cong \operatorname{Tr} D^j P$, where Tr denotes the transpose and D the ordinary duality. M. Auslander and S. Smalø gave a definition of preprojective modules for arbitrary artin alegbras, which in the hereditary case coincides with the previous one. They proved [2] that for any artin algebra there exist unique sets of isomorphism classes of indecomposable modules $\underline{\underline{P}}_i$, $i \in N \cup \{\infty\}$ such that:

(i) $\underline{\underline{P}}_0$ consists of the indecomposable projective Λ-modules.

(ii) $\underline{\underline{P}}_n$ is finite for each $n < \infty$ and is minimal with respect to the property that each indecomposable module not in $\bigcup_{i=0}^{n-1} \underline{\underline{P}}_i$ is a quotient of a direct sum of modules from $\underline{\underline{P}}_n$.

(iii) $\underline{\underline{P}}_\infty$ is the collection of indecomposable modules not in $\bigcup_{i<\infty} \underline{\underline{P}}_i$. A module is called *preprojective* if it is in $\bigcup_{i<\infty} \underline{\underline{P}}_i$.

One of the basic tools in describing preprojective partitions is irreducible maps defined by M. Auslander and I. Reiten in [1]. A nonzero map $f\colon M \to N$ is irreducible if f is neither a splittable monomorphism nor a splittable epimorphism and, given any factorization of f,

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "g"'] & & N \\ & L \arrow[ur, "h"'] & \end{tikzcd}$$

then either g is a splittable monomorphism or h is a splittable epimorphism.

The following result about preprojective modules is true for arbitrary artin algebras [3].

(a) $A \in \underline{\underline{P}}_0$ if and only if A is an indecomposable projective module.

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(b) $A \in \underline{\underline{P}}_1$ if and only if $A \cong \operatorname{Tr} DB$ for some summand B of the radical of Λ as left module.

(c) If $A \in \underline{\underline{P}}_n$, then there exist a B in $\underline{\underline{P}}_j$ for some $j < n$ and an irreducible morphism $B \to A$.

This result gives a complete description of the modules in $\underline{\underline{P}}_0$ and $\underline{\underline{P}}_1$, and a way to find $\underline{\underline{P}}_n$, $n < \infty$, when all the $\underline{\underline{P}}_j$, $j < n$ are given, which is described in the same paper. First look at the set $\underline{\underline{A}}_n$ of all indecomposable modules A such that there exists an irreducible morphism $f\colon B \to A$ for some B in $\bigcup_{i<n} \underline{\underline{P}}_i$. $\bigcup_{i<n} \underline{\underline{P}}_i$ is finite so the set $\underline{\underline{A}}_n$ will also be finite [**1**]. Further it is possible to prove that $\underline{\underline{A}}_n$ is a cover for $\operatorname{ind} \Lambda - \bigcup_{i<n} \underline{\underline{P}}_i$ [**2**], and therefore it contains a cover with a minimal number of elements which then is the unique minimal cover $\underline{\underline{P}}_n$ of $\operatorname{ind} \Lambda - \bigcup_{i<n} \underline{\underline{P}}_i$. To determine the minimal cover one can start with any A in $\underline{\underline{A}}_n$ and see if $\underline{\underline{A}}_n - \{A\}$ is a cover for $\underline{\underline{A}}_n$. A is then in the minimal cover $\underline{\underline{P}}_n$ if and only if $\underline{\underline{A}}_n - \{A\}$ is not a cover for $\underline{\underline{A}}_n$. Hence, by going through all modules in $\underline{\underline{A}}_n$ in an arbitrary order one can find the members of $\underline{\underline{P}}_n$.

If Λ is a hereditary artin algebra, instead of condition (c), we give necessary and sufficient conditions for a module A to be in $\underline{\underline{P}}_n$: A is in $\underline{\underline{P}}_n$ if and only if it is not in $\underline{\underline{P}}_{n-1}$ and there is an irreducible map $B \to A$, where B is in $\underline{\underline{P}}_{n-1}$. This way the construction described before becomes much simpler: it is enough to consider the set $\underline{\underline{A}}_n$ and exclude all modules that are in $\underline{\underline{P}}_{n-1}$ in order to obtain $\underline{\underline{P}}_n$, which means just by looking at the modules and without considering maps and possible covers we can determine which modules are in $\underline{\underline{P}}_n$.

Dually, one can define preinjective partitions and the dual statements hold. D. Zacharia applied these results to obtain some interesting results about preprojective and preinjective partitions for hereditary and stably equivalent to hereditary artin algebras [**4**].

PROPOSITION 1. *Let Λ be a hereditary artin algebra. Then Y is in $\underline{\underline{P}}_n$ if and only if both of the following conditions hold*:

(a) *For every irreducible map $f\colon X \to Y$ with X indecomposable, X is in $\underline{\underline{P}}_{n-1} \cup \underline{\underline{P}}_n$, and*

(b) *there is an indecomposable X_{n-1} in $\underline{\underline{P}}_{n-1}$ and an irreducible map $f\colon X_{n-1} \to Y$.*

PROOF. (a) The proof will be by induction on n. If $n = 0$, it is clear since Λ is hereditary. Suppose it is true for $j \leqslant n$. Let A_{n+1} be in $\underline{\underline{P}}_{n+1}$ and suppose $X \to A_{n+1}$ is irreducible, and X is indecomposable. Suppose X is in $\underline{\underline{P}}_t$ for some $t \leqslant n-1$. Consider an almost split sequence

$$0 \to C \to X \amalg B \to A_{n+1} \to 0. \tag{1}$$

Let B'_{n-1} be a summand of B containing all summands of B which are in $\bigcup_{k \leqslant n-1} \underline{\underline{P}}_k$, and B' is determined by $B = B'_{n-1} \amalg B'$. Then B' has no summands from $\bigcup_{k \leqslant n-1} \underline{\underline{P}}_k$. So (1) looks like:

$$0 \to C \to X \amalg B'_{n-1} \amalg B' \to A_{n+1} \to 0.$$

Since $\underline{\underline{P}}_n$ is a generating set for $\Lambda - \bigcup_{k<n}\underline{\underline{P}}_n$, there exists an epimorphism f: $\amalg Y_n \to A_{n+1}$ with all Y_n in $\underline{\underline{P}}_n$.

CLAIM. If X is in $\bigcup_{k<n}\underline{\underline{P}}_k$, then $\operatorname{Hom}_\Lambda(Y_n, X) = 0$.

PROOF OF THE CLAIM. Suppose X is in $\underline{\underline{P}}_{n-1}$. Call it X_{n-1}. Let f be a nonzero map $Y_n \to X_{n-1}$. We may assume that X_{n-1} is not projective. Consider an almost split sequence $0 \to K \to \amalg Z_k \to X_{n-1} \to 0$. By induction hypothesis all Z_k's are in $\underline{\underline{P}}_{n-2} \cup \underline{\underline{P}}_{n-1}$. So f factors through some Z_k which is either in $\underline{\underline{P}}_{n-2}$ or $\underline{\underline{P}}_{n-1}$. Clearly, if Z_k is in $\underline{\underline{P}}_{n-1}$, then irreducible map $Z_k \to X_{n-1}$ is a monomorphism. Applying the same argument either we reach some module in $\bigcup_{k<n}\underline{\underline{P}}_k$ which is isomorphic to Y_n, which is impossible since Y_n is in $\underline{\underline{P}}_n$, or we have an infinite chain of irreducible monomorphisms in the same $\underline{\underline{P}}_k$, which is again impossible, or we have a map from Y_n to a projective, which is impossible since Λ is hereditary. This finishes proof of the claim.

Therefore $\operatorname{Hom}_\Lambda(Y_n, X) = 0$ and $\operatorname{Hom}_\Lambda(Y_n, B'_{n-1}) = 0$. So f factors through B'. Thus $B' \to A_{n+1}$ is an epimorphism, which implies that $C \to X$ is an epimorphism. Since X is in $\underline{\underline{P}}_t$, the module C cannot be in $\underline{\underline{P}}_t$, otherwise $\underline{\underline{P}}_t$ would not be a minimal generating set, and by the induction hypothesis C cannot be in $\bigcup_{j<t-1}\underline{\underline{P}}_j$. Thus C is in $\underline{\underline{P}}_{t-1}$. Then we will denote C by C_{t-1}. (Similarly we will write $X = X_t$ since X is in $\underline{\underline{P}}_t$.) So the sequence (1) is

$$\begin{tikzcd}
 & & X_t \arrow[drr] & & & \\
0 \arrow[r] & C_{t-1} \arrow[ur] \arrow[rr, two heads] \arrow[dr] & & B'_{n-1} \arrow[r] & A_{n+1} \arrow[r] & 0 \\
 & & B' \arrow[urr] & & &
\end{tikzcd}$$

Not all summands of B' can be in $\bigcup_{k\geqslant n+1}\underline{\underline{P}}_k$, otherwise A_{n+1} would not be in the minimal generating set $\underline{\underline{P}}_{n+1}$. Therefore there exists a summand of B' which is in $\underline{\underline{P}}_n$. Call it B_n. But then from the above sequence we have an irreducible map $C_{t-1} \to B_n$, which is a contradiction to the induction hypothesis, since $t - 1 < n - 1$.

($*$) Therefore if there is an irreducible map $X_t \to A_{n+1}$, where X_t is in $\underline{\underline{P}}_t$, then $t \geqslant n$.

Now suppose there is an irreducible map $X_t \to A_{n+1}$ with X_t in $\underline{\underline{P}}_t$ and $t > n + 1$. Consider an almost split sequence

$$0 \to C \to X_t \amalg B \to A_{n+1} \to 0.$$

By condition (c) which was stated for arbitrary artin algebras there exists a summand of B which is in $\underline{\underline{P}}_k$ with $k \leqslant n$. But by ($*$) $k \geqslant n$, so $k = n$. So the above sequence looks like $0 \to C \to X_t \amalg B' \amalg B_n \to A_{n+1} \to 0$ where $B \cong B' \amalg B_n$ and B_n is in $\underline{\underline{P}}_n$. Since there is an irreducible map $C \to B_n$, by the induction hypothesis, C is in $\underline{\underline{P}}_s$

with $n-1 \leqslant s \leqslant n$. We denote C by C_s. Consider now an almost split sequence for X_t:

$$\begin{tikzcd} & & D \arrow[dr] & & \\ 0 \arrow[r] & E \arrow[ur] \arrow[dr] & & X_t \arrow[r] & 0 \\ & & C_s \arrow[ur] & & \end{tikzcd}$$

Since $t \geqslant n+2$, there exists an epimorphism $f\colon Y \to X_t$ with all summands of Y in $\underline{\underline{P}}_{n+1}$. Let D_n^1 be a summand of D containing all summands which are in $\bigcup_{k \leqslant n} \underline{\underline{P}}_k$. Let $D = D^1 \amalg D_n^1$. Then $\mathrm{Hom}_\Lambda(Y, D_n^1) = 0$, and also $\mathrm{Hom}_\Lambda(Y, C_s) = 0$. So f factors through D^1 and $D^1 \to X_t$ is an epimorphism. Thus the previous sequence looks like

$$\begin{tikzcd} & & D^1 \arrow[dr] & & \\ 0 \arrow[r] & E \arrow[ur] \arrow[r, two heads] \arrow[dr] & D_n^1 \arrow[r] & X_t \arrow[r] & 0 \\ & & C_s \arrow[ur] & & \end{tikzcd}$$

So there exists a summand D_{j_1} of D^1, with D_{j_1} in $\underline{\underline{P}}_{j_1}$ with $j_1 < t$ since X_t is in $\underline{\underline{P}}_t$ which is a minimal generating set. Also $n+1 \leqslant j_1$ since D_{j_1} is a summand of D^1. So $n+1 \leqslant j_1 < t$. Since the map $E \to C_s$ is an epimorphism, by the induction hypothesis E is in $\underline{\underline{P}}_{s-1}$. Denote $E = E_{s-1}$. Then there is an irreducible map $E_{s-1} \to D_{j_1}$. If $j_1 = n+1$, then by $(*)$ $s-1 \geqslant n$, which is a contradiction since $s \leqslant n$. Therefore $n+1 < j_1 < t$.

Consider now an almost split sequence:

$$\begin{tikzcd} & & D^2 \arrow[dr] & & \\ 0 \arrow[r] & E_{s-2} \arrow[ur] \arrow[r, two heads] \arrow[dr] & D_n^2 \arrow[r] & D_{j_1} \arrow[r] & 0 \\ & & E_{s-1} \arrow[ur] & & \end{tikzcd}$$

where D^2 has no summand from $\bigcup_{j \leqslant n} \underline{\underline{P}}_j$, and D_n^2 contains all such summands except for E_{s-1}. Then by the argument as before, $D^2 \to D_{j_1}$ is an epimorphism, $E_{s-2} \to E_{s-1}$ is an epimorphism, and E_{s-2} is in $\underline{\underline{P}}_{s-2}$. Using the same argument that we used to get D_{j_1}, we can show that there is a summand D_{j_2} of D^2 with D_{j_2} in $\underline{\underline{P}}_{j_2}$ and $n+1 < j_2 < j_1$. This way we obtain a chain of decreasing positive integers, which are all larger than $n+1$, giving a contradiction.

(b) This now follows from part (a) and the condition (c) stated for arbitrary artin algebras. $\square$

COROLLARY 2. *Let Λ be a hereditary artin algebra and let $\underline{\underline{A}}_n = \{X$ in $\operatorname{ind}\Lambda \mid$ there exists an irreducible map $A_{n-1} \to X$ with A_{n-1} in $\underline{\underline{P}}_{n-1}\}$. Then $\underline{\underline{P}}_n = \underline{\underline{A}}_n - \underline{\underline{P}}_{n-1}$, for $n \geq 1$.*

PROOF. Suppose Y is in $\underline{\underline{P}}_n$. Then if $X \to Y$ is irreducible, X is in $\underline{\underline{P}}_n \cup \underline{\underline{P}}_{n-1}$ by Proposition 1(a), and by (b) there exist X_0 in $\underline{\underline{P}}_{n-1}$ and an irreducible map $X_0 \to Y$. So Y is in $\underline{\underline{A}}_n$, and since $\underline{\underline{P}}_i$'s are disjoint, Y is in $\underline{\underline{A}}_n - \underline{\underline{P}}_{n-1}$. Now suppose Y is in $\underline{\underline{A}}_n - \underline{\underline{P}}_{n-1}$. Then there exists an irreducible map $A_{n-1} \to Y$ with A_{n-1} in $\underline{\underline{P}}_{n-1}$. Then by Proposition 1, Y is in $\underline{\underline{P}}_{n-1} \cup \underline{\underline{P}}_n$, and therefore Y is in $\underline{\underline{P}}_n$. □

COROLLARY 3. *Suppose X is in $\underline{\underline{P}}_n$. Then there exists a chain of irreducible maps $X_0 \to X_1 \to \cdots \to X_j = X$ with X_0 projective, and the shortest such chain is of length n.*

PROOF. If X is in $\underline{\underline{P}}_n$, then there exist a module X_{n-1} in $\underline{\underline{P}}_{n-1}$ and an irreducible map $X_{n-1} \to X$. Applying the same argument, we obtain a chain $X_0 \to X_1 \to \cdots \to X_{n-1} \to X$, with X_k in $\underline{\underline{P}}_k$ for all $0 \leq k \leq n-1$. Let $X_0' \to X_1' \to \cdots \to X_{j-1}' \to X_j' = X$ be the shortest chain of irreducible maps with X_0' in $\underline{\underline{P}}_0$. Then by Proposition 1 if X_s' is in $\underline{\underline{P}}_s$, the module X_{s+1}' is in $\underline{\underline{P}}_s \cup \underline{\underline{P}}_{s+1}$. So we have X_1' is in $\underline{\underline{P}}_0 \cup \underline{\underline{P}}_1$, and X_k' is in $\bigcup_{i<k} \underline{\underline{P}}_i$ for all $k \leq j$. So X_j' is in $\bigcup_{i \leq j} \underline{\underline{P}}_i$, and since $X_j' = X$ which is in $\underline{\underline{P}}_n$, it follows that $n \leq j$. But j was chosen to be minimal, so $n = j$. □

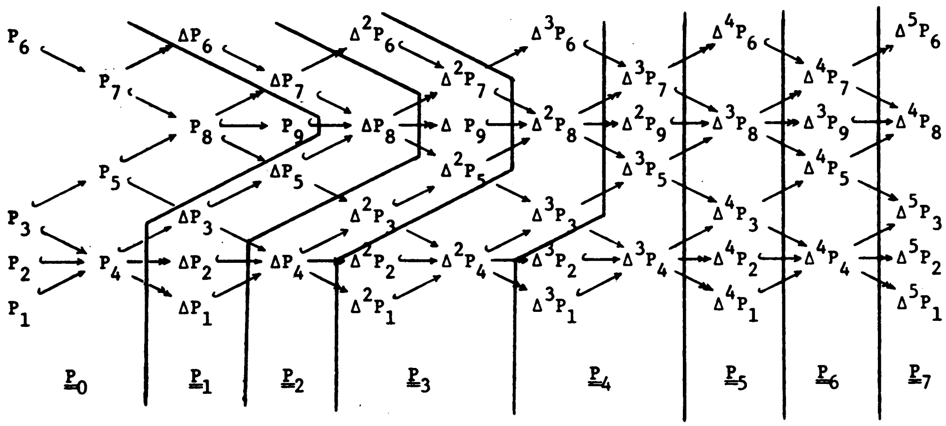
To illustrate the results, we give an example of a hereditary artin algebra, the first few classes of the partition $\bigcup \underline{\underline{P}}_i$, and the irreducible maps between the modules in these classes: let k be a field Λ the subring of 8×8 lower triangular matrix ring, given by

$$\Lambda = \begin{array}{l} \left(\begin{array}{l} k \\ 0\,k \\ 0\,0\,k \\ k\,k\,k\,k \\ 0\,0\,k\,0\,k \\ 0\,0\,0\,0\,0\,k \\ 0\,0\,0\,0\,0\,k\,k \\ 0\,0\,k\,0\,k\,k\,k\,k \\ 0\,0\,k\,0\,k\,k\,k\,k\,k \end{array}\right) \end{array} \begin{array}{l} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{array}$$

where the P_i's are indecomposable right projective Λ-modules. This ring corresponds to the ring given by the diagram

$$\begin{tikzcd} & & & & \cdot 9 \arrow[d] & & \\ \cdot 1 & \cdot 4 \arrow[l] \arrow[r] \arrow[d] & \cdot 3 & \cdot 5 \arrow[l] & \cdot 8 \arrow[l] \arrow[r] & \cdot 7 \arrow[r] & 6 \cdot \\ & \cdot 2 & & & & & \end{tikzcd}$$

This ring is of infinite representation type, and a part of the Auslander-Reiten graph looks like



where Δ denotes $\operatorname{Tr} D$.

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