ON THE UNIFORM ASYMPTOTIC STABILITY IN FUNCTIOTAL DIFFERENTIAL EQUATIONS

L. Z. WEN

ABSTRACT. We consider a system of functional differential equations $x'(t) = F(t, x(t))$ and obtain conditions on a Liapunov functional to insure the uniform asymptotic stability of the zero solution.

1. Introduction. Following the work of Yoshizawa [2], Burton [1] obtained sufficient conditions of the uniform asymptotic stability in the retarded functional differential equation $x'(t) = F(t, x(t))$ on a Liapunov functional. He showed that it is not necessary to require $F(t, x(t))$ bounded for $x(t)$ bounded. Now we use the Razumikhin condition so that it is not necessary to require $V'(t, x(t)) = -W(|x(t)|)$ for all $t \geq 0$. This work generalized Burton’s result.

For $x \in \mathbb{R}^n$, let $|x|$ be $\max_{1 \leq i \leq n} |x_i|$. Given $h > 0$, let $C$ denote the space of continuous functions from $[-h, 0]$ into $\mathbb{R}^n$ and for $\phi \in C$, $\|\phi\| = \sup_{-h \leq \theta \leq 0} |\phi(\theta)|$. For $\phi \in C_H = \{\phi: \phi \in C, \|\phi\| \leq H\}$, let

$$\|\phi\| = \left( \sum_{i=1}^n \int_{-h}^0 \phi_i^2(s) \, ds \right)^{1/2},$$

where $\phi_i$ are the components of $\phi$.

For $t_0 \in \mathbb{R}$, $A > 0$, $t \in [t_0, t_0 + A]$ and a continuous function $x$ from $[t_0 - h, t_0 + A]$ into $\mathbb{R}^n$, let $x_\theta \in C$ be defined by $x_\theta(0) = x(t + \theta)$, $\theta \in [-h, 0]$.

2. Uniform asymptotic stability.

LEMMA. Let $F$ be a family of continuous functions $f: [a, b] \to [0, 1]$ and $W: [0, \infty) \to [0, \infty)$ be a continuous nondecreasing function, and $W(s) > 0$ if $s > 0$. If there exists $\alpha > 0$ with $\int_a^b f(t) \, dt \geq \alpha$ for any $f \in F$ then there exists $\beta > 0$ with $\int_0^1 W(f(t)) \, dt \geq \beta$.

PROOF. For any $f \in F$, let $E = \{t: f(t) \geq \alpha/2(b - a), a \leq t \leq b\}$ and $m(E)$ be the measure of $E$. If $m(E) < \alpha/2$, then

$$\alpha \leq \int_a^b f(t) \, dt = \int_E f(t) \, dt + \int_{[a, b] \setminus E} f(t) \, dt < \alpha/2 + \alpha/2 = \alpha,$$

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a contradiction. Hence \( m(E) > \alpha/2 \) and
\[
\int_a^b W(f(t)) \, dt \geq \int_E W(f(t)) \, dt \geq \int_E W(\alpha/2(b - a)) \, dt \geq W(\alpha/2(b - a))\frac{\alpha}{2} \overset{\text{def}}{=} \beta.
\]

This completes the proof.

We consider the retarded functional differential equation
\[
(1) \quad x'(t) = F(t, x_t),
\]
where \( x'(t) \) is the right-hand derivative of \( x(t) \) and \( F(t, x_t) \) a continuous function from \( R \times C_H \) into \( R^n \), \( F(t, 0) = 0 \). For continuation of solution, we suppose that \( F \) takes closed bounded sets of \( R \times C_H \) into closed bounded sets of \( R^n \).

Denote by \( x(t_0, \phi) \) a solution of (1) with initial condition \( \phi \in C_H \) where \( x(t_0, \phi) = \phi \) and we denote by \( x(t) = x(t, t_0, \phi) \) the value of \( x(t_0, \phi) \) at \( t \).

Let \( V(t, \phi) \) be a continuous nonnegative functional defined in \([0, \infty) \times C_H \). The upper right-hand derivative of \( V \) along solution of (1) is defined to be
\[
V'(t, x_t(t_0, \phi)) = \lim_{\delta \to 0^+} \frac{V(t + \delta, x_{t+\delta}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{\delta}.
\]

We suppose that \( V'(t, x_t) \) exists.

Let \( W_1, W_2, W_3, W \) be continuous nondecreasing functions and \( P \) be a continuous function from \([0, \infty) \) into \([0, \infty) \) with \( W_1(r) > 0, W_2(r) > 0, P(r) > r \) if \( r > 0 \) and \( W(0) = 0 \).

**Theorem.** Suppose there are functions \( W_1, W_2, W_3, W_\) as above, which also satisfy the following conditions:

(i) \( W_1(|\phi(0)|) \leq V(t, |\phi|) \leq W_2(|\phi(0)|) + W_3(|\phi|) \) for any \( \phi \in C_H \).

(ii) For any \( t_0 \geq 0 \) and any \( \phi \in C_H \)
\[
V'(t, x_t(t_0, \phi)) < 0 \quad \text{if} \quad V(t, x_t(t_0, \phi)) \leq W_2(|\phi|) + W_3(|\phi|) \quad (t \leq t_0 + h),
\]
and
\[
V'(t, x_t(t_0, \phi)) \leq -W(|x(t, t_0, \phi)|) \quad \text{if} \quad P(V(t, x_t(t_0, \phi))) > V(\xi, x_{\xi}(t_0, \phi))
\]
\[
(t \geq t_0 + h; t - h < \xi < t).
\]

Then the zero solution of (1) is uniformly asymptotically stable.

**Proof.** We first prove the uniform stability. Given \( \epsilon > 0 \ (\epsilon < H, W_1(\epsilon) < H) \), choose \( \delta > 0 \) such that \( \delta < \epsilon, W_2(\delta) < W_1(\epsilon)/2, \) and \( W_3(\delta/\sqrt{\epsilon}) < W_1(\epsilon)/2 \). Let \( t_0 \geq 0 \) and \( ||\phi|| < \delta \). We shall show that
\[
(2) \quad V(t, x_t(t_0, \phi)) < W_1(\epsilon) \quad (t \geq t_0).
\]

Obviously,
\[
V(t_0, \phi) \leq W_2(|\phi(0)|) + W_3(||\phi||) \leq W_2(\delta) + W_3(\delta/\sqrt{\epsilon}) < W_1(\epsilon).
\]

For each \( t \in [t_0, t_0 + h] \), if \( V(t, x_t) < W_2(||\phi||) + W_3(||\phi||) \), then \( V(t, x_t) < W_1(\epsilon) \), if \( V(t, x_t) = W_2(||\phi||) + W_3(||\phi||) \), from condition (ii) we get \( V(t + \Delta t, x_{t+\Delta t}) \leq W_2(||\phi||) + W_3(||\phi||) \) for all sufficiently small \( \Delta t > 0 \). It implies that \( V(t, x_t) < W_1(\epsilon) \)
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for all $t \in [t_0, t_0 + h)$. Thus, if (2) fails, then there exists $t_1 > t_0 + h$ such that

$$V(t_1, x_{t_1}) = W_1(\epsilon) \quad V(t, x_t) \leq W_1(\epsilon) \quad (t \leq t_1).$$

Let $d = \inf_{W_2(\|\phi\|) + W_3(\|\phi\|) \leq W_1(\epsilon)} [P(r) - r]$. Obviously, there exists $T \in (t_0 + h, t_1)$ such that

(a) $W_2(\|\phi\|) + W_3(\|\phi\|) \leq W_1(\epsilon) - \frac{1}{\epsilon}d < V(T, x_T) < W_1(\epsilon)$, where $\epsilon > 1$,

(b) $V'(T, x_T) > 0$.

From (a),

$$P(V(T, x_T)) = V(T, x_T) + d > W_1(\epsilon) + \left(1 - \frac{1}{\epsilon}\right)d > V(\xi, x_\xi) \quad (t_0 < \xi < T).$$

From condition (ii), we have $V'(T, x_T) \leq -W(\|x(T)\|) \leq 0$, which contradicts (b). Hence, (2) holds.

By (2) and condition (i), we get $|x(t)| < \epsilon$ for $t \geq t_0$. Since $\delta$ is independent of $t_0$, this proves the uniform stability.

Next, we prove the uniform asymptotic stability. For $H^* = \min[H, 1]$ choose $d > 0$ such that $|x(t, t_0, \phi)| < H^*$ for $t \geq t_0$, if $t_0 > 0$ and $\|\phi\| < \delta$. From condition (i), we have

$$V(t, x_t) < W_1(\epsilon) + (N - 1)d.$$ (3)

If not, then

$$V(t, x_t) \geq W_1(\epsilon) + (N - 1)d \quad (t \geq t_0 + h),$$

and

$$P(V(t, x_t)) = V(t, x_t) + d \geq W_1(\epsilon) + N\bar{d} \geq B > V(\xi, x_\xi) \quad (t_0 \leq \xi \leq t).$$

From (ii) we have $V'(t, x_t) \leq -W(\|x(t)\|) (t \geq t_0 + h)$; it follows that

$$V(t, x_t) < B - \int_{t_0 + h}^t W(\|x(s)\|) ds.$$ (4)

If $V(t, x_t) \geq W_1(\epsilon)$, then

$$W_2(\|x(t)\|) + W_3(\|x(t)\|) > V(t, x_t) > W_1(\epsilon).$$

Therefore, either $W_2(\|x(t)\|) \geq W_1(\epsilon)/2$ or $W_3(\|x(t)\|) \geq W_1(\epsilon)/2$. Let $E_1 = \{t: W_2(\|x(t)\|) \geq W_1(\epsilon)/2, t \geq t_0\}$ and $E_2 = [t_0, \infty) - E_1$. If $t \in E_1$, then there exists a constant $a > 0$ with $\|x(t)\| > a$. If $t \in E_2$, then there exists a constant $b > 0$ with $|x(t)| > b$. In case $t \in E_1$, we have

$$\sum_{i=1}^n \int_{-h}^0 x_i^2(t + \theta) \, d\theta \geq a^2,$$
then
\[ \int_{t-h}^{t} \frac{1}{n} \sum_{i=1}^{n} x_i^2(s) \, ds \geq \frac{a^2}{n} = \alpha. \]

Since \(|x(t)| < 1\), we have
\[ |x(t)| = \max_i |x_i(t)| \geq \frac{1}{n} \sum_{i=1}^{n} x_i^2(t). \]

Then from the Lemma, there exists \( \beta > 0 \) such that
\[ \int_{t-h}^{t} W(|x(s)|) \, ds > \int_{t-h}^{t} W \left( \frac{1}{n} \sum_{i=1}^{n} x_i^2(s) \right) \, ds > \beta. \]

Let \( K \) be the positive integer satisfying \( K > B > (K-1) \) and \( T_i = t_0 + (K+1)h + 2B/W(b) \), we have either
\( \text{(a) } m(E_1 \cap [t_0 + h, T_i]) > Kh \) or
\( \text{(b) } m(E_2 \cap [t_0 + h, T_i]) > 2B/W(b). \)

If (a) holds, then in \( E_1 \cap [t_0 + h, T_i] \) there exist \( K \) points \( t_1 < t_2 < \cdots < t_k \) satisfying \( t_j - t_{j-1} > h \) \((j = 2, 3, \ldots, K)\). From (4) and (5), we have
\[ V(T_i, x_{T_i}) < B - \int_{t_0 + h}^{T_i} W(|x(s)|) \, ds \]
\[ \leq B - \sum_{j=1}^{k} \int_{t_j-h}^{t_j} W \left( \frac{1}{n} \sum_{i=1}^{n} x_i^2(s) \right) \, ds \leq B - k\beta < 0. \]

If (b) holds, from (4) we have
\[ V(T_i, x_{T_i}) < B - \int_{E_2 \cap [t_0 + h, T_i]} W(b) \, ds = B - W(b)m(E_2 \cap [t_0 + h, T_i]) < 0. \]

Thus either (a) or (b) implies \( V(T_i, x_{T_i}) < 0 \), a contradiction to \( V(t, x_{t_i}) > 0 \). Hence (3) holds.

In the following, we will show that
\[ V(t, x_t(t_0, \varphi)) < W_1(e) + (N-1)d \quad \text{for all } t > T_1. \]

If (6) is not true, then there exists \( \sigma > T_1 \) such that \( V(\sigma, x_{\sigma}) < W_1(e) + (N-1)d \) and
\( (A) B - W_2(H^*) - W_2(H^* \sqrt{nh}) > W_1(e) + (N-1)d - V(\sigma, x_{\sigma}), \)
\( (B) V'(\sigma, x_{\sigma}) > 0. \)

From (A), we get
\[ P(V(\sigma, x_{\sigma})) > V(\sigma, x_{\sigma}) + \overline{d} \]
\[ \geq W_1(e) + (N-1)d - B + W_2(H^*) + W_3(H^* \sqrt{nh}) + \overline{d} \]
\[ = W_1(e) + N\overline{d} - B + W_2(H^*) + W_3(H^* \sqrt{nh}) \]
\[ \geq W_2(H^*) + W_3(H^* \sqrt{nh}) \geq V(\xi, x_{\xi}) \quad (t_0 \leq \xi \leq \sigma). \]
From condition (ii) we have $V'(\sigma, x_\sigma) \leq -W(|x(\sigma)|) \leq 0$, which contradicts (B). Therefore, (6) holds.

Similarly, there exists $T_2, T_3, \ldots, T_N$ such that

$$V(t, x(t_0, \phi)) < W_1(\epsilon) + (N - k)\overline{d} \quad \text{for } t \geq T_k, \quad k = 2, 3, \ldots, N.$$  

Then $V(t, x(t_0, \phi)) < W_1(\epsilon)$ for all $t \geq T_N$. From condition (i) we have $|x(t)| < \epsilon$ for all $t \geq T_N$, where

$$T_N = t_0 + N((k + 1)h + 2B/W(b)).$$  

Since $N((k + 1)h + 2B/W(b))$ is independent of $t_0$, we have completed the proof of the theorem.

**Example.** Consider the equation

$$x'(t) = -a(t)x(t) + b(t)x(t - h)$$

where $a(t)$ and $b(t)$ are continuous functions, $0 < a \leq a(t) < \infty$, $|b(t)| \leq b < \mu a$, $0 < \mu < 1$.

One can choose $V(t, x) = \frac{1}{2}x^2(t)$, $W_1(|x(t)|) = \frac{1}{4}x^2(t)$, $W_2(|x(t)|) = x^2(t)$, $W_3(||x||) = ||x||^2$ and $P(s) = qs, q > 1$.

For $t \in [t_0, t_0 + h)$, if $V(t, x) = W_2(||\phi||) + W_3(||\phi||)$, that is $\frac{1}{2}x^2(t) = ||\phi||^2 + ||\phi||^2$. Then

$$V'(t, x) = x(t)x'(t) = -a(t)x^2(t) + b(t)x(t)x(t - h)$$

$$\leq -ax^2(t) + b\left[x^2(t) + x^2(t - h)\right]$$

$$\leq -\left(a - \frac{b}{2}\right)x^2(t) + \frac{b}{2}||\phi||^2 = -\left(2a - \frac{3b}{2}\right)||\phi||^2 - (2a - b)||\phi||^2 < 0.$$  

For $t \in [t_0 + h, \infty)$ if $P(V(t, x)) > V(\xi, x_\xi)$ $(t - h \leq \xi \leq t)$, that is $qx^2(t) > x^2(\xi) (t - h \leq \xi \leq t)$, then $qx^2(t) > x^2(t - h)$.

$$V'(t, x) \leq -\left(a - \frac{b}{2}\right)x^2(t) + \frac{b}{2}x^2(t - h)$$

$$\leq -\left(a - \frac{b}{2}\right)x^2(t) + \frac{b}{2}qx^2(t) = -\left(a - b\left(\frac{1 + q}{2}\right)\right)x^2(t).$$

If we choose $q = 2/\mu - 1$, then $a - b((1 + q)/2) > 0$. Let

$$W(|x(t)|) = (a - b((1 + q)/2))x^2(t).$$

We can see that the conditions of the Theorem are satisfied. Therefore, the zero solution of (7) is uniformly asymptotically stable.

**References**


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DEPARTMENT OF MATHEMATICS, HUNAN UNIVERSITY, CHANGSHA, HUNAN, PEOPLE’S REPUBLIC OF 
CHINA