ISOMETRIES OF $\mathcal{A}_c(K)$

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Abstract. We completely describe isometries of $\mathcal{A}_c(K)$, when $K$ is a compact
Choquet simplex, using facially continuous functions on the extreme boundary.

1. Introduction. Let $K$ be a compact convex set in a locally convex space and
denote by $E(K)$ the set of extreme points of $K$ and by $\mathcal{A}_c(K)$ the continuous
complex-valued affine functions on $K$, equipped with the supremum norm.

We first describe a class of isometries for $\mathcal{A}_c(K)$ when $K$ is any compact convex
set and give a sufficient condition on an isometry, in terms of facially continuous
functions on $E(K)$, so that the isometry in question is in the prescribed class and
then deduce that when $K$ is a Choquet simplex, the class of isometries considered,
completely describes the isometries of $\mathcal{A}_c(K)$.

2. Notations and definitions. For the concepts and results of convexity theory used
here we cite [1].

A set $D \subset E(K)$ is said to be facially closed if there exists a closed split face $F$ of
$K$ such that $E(F) = D$. The sets $D$ form the closed sets of a topology on $E(K)$
called the facial topology.

Let $C$ denote the complex plane and $\Gamma$, the unit circle in $C$. For a probability
measure $\mu$, let $r(\mu)$ denote the resultant of $\mu$ and Supp $\mu$ denote the topological
support of $\mu$.

3. Description of isometries. Following the notations of [1], we denote by $Z(\mathcal{A}_c(K))$
the set of elements $b \in \mathcal{A}_c(K)$ such that for every $a \in \mathcal{A}_c(K)$ there exists $c \in \mathcal{A}_c(K)$
satisfying

$$c(x) = a(x) \cdot b(x) \quad \forall x \in E(K).$$

Since for any $b \in Z(\mathcal{A}_c(K))$, real and imaginary parts of $b$ are in $Z(A(K))$, using
Corollary II.7.4 and Theorem II.7.10 of [1], we can easily see that for $b \in \mathcal{A}_c(K)$, $b$
is in $Z(\mathcal{A}_c(K))$ if and only if $b \mid E(K) \to C$ is continuous in the facial topology.

Let $Q: K \to K$ be an onto affine homeomorphism and let $a_0 \in Z(\mathcal{A}_c(K))$ be such
that $|a_0| = 1$ on $E(K)$. Define $\Phi: \mathcal{A}_c(K) \to \mathcal{A}_c(K)$ by $\Phi(a) = c$, where $c$ is the
unique element of $\mathcal{A}_c(K)$ such that $c(x) = a(Q(x)) \cdot a_0(x) \forall x \in E(K)$.

It is easy to see that $\Phi$ is an onto isometry and $\Phi(1) = a_0$.
Theorem 3.1. Let \( \Phi : A_c(K) \rightarrow A_c(K) \) be any onto isometry. Assume
\[ \Phi(1) \in Z(A_c(K)). \]
Then there exists an affine homeomorphism \( Q \) from \( K \) onto \( K \) such that
\[ \Phi(a)(x) = a(Q(x)) \Phi(1)(x) \quad \forall x \in E(K). \]

Proof. Define \( \delta : K \rightarrow A(K)^* \) by \( \delta(x)(a) = a(x) \forall a \in A_c(K) \) and \( x \in K \). It is well known that \( \delta \) is an affine homeomorphism of \( K \) onto \( \{ f \in A_c(K)^* : \| f \| = \| f(1) \| = 1 \} \), with \( w^* \)-topology. Since \( \Phi^* : A_c(K)^* \rightarrow A_c(K)^* \) is an onto isometry and a \( w^* \)-homeomorphism it is easy to see that \( \Phi^*(\delta(E(K))) \subset \Gamma \cdot \delta(E(K)). \)

Let \( x \in E(K) \). Since \( A_c(K) \) separates points of \( K \) and \( 1 \in A_c(K) \), there exist unique \( x' \in E(K) \) and \( t \in \Gamma \), such that \( \Phi^*(\delta(x)) = t \cdot \delta(x') \). Moreover
\[ (*) \quad \Phi^*(\delta(x))(1) = \delta(x)(\Phi(1)) = \Phi(1)(x) = t. \]
Hence \( \Phi(1) \) is of modulus 1 on \( E(K) \). Let \( \Phi(1) = u + iv, u, v \in A(K) \) (real-valued functions in \( A_c(K) \)). Then since \( Z(A_c(K)) \) is selfadjoint, \( \Phi(1) = u - iv \) is in \( Z(A_c(K)) \). Define now \( T : A_c(K) \rightarrow A_c(K) \) by
\[ T(a)(x) = \Phi(a)(x) \cdot \Phi(1)(x) \quad \forall x \in E(K). \]
Since \( |\Phi(1)| = 1 \) on \( E(K) \), it follows from the remarks in the beginning of this section that \( T \) is a well-defined, onto isometry. Moreover, \( T(1) = 1 \). It is easy to see that \( T^* \) maps \( \delta(K) \) onto \( \delta(K) \) and \( Q = \delta^{-1} \circ T^* \circ \delta \) is an affine homeomorphism of \( K \) onto \( K \). That \( \Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x) \forall x \in E(K) \) follows from \( (*) \) and the definition of \( T \).

Definition (Effros). Say a closed set \( D \subset K \) is a dilated set if for any maximal measure \( \mu \) with \( r(\mu) \in D \), \( \text{Supp } \mu \subset D \).

Proposition 3.2. Let \( K \) be a compact Choquet simplex and let \( a_0 \in A_c(K) \) and \( | a_0 | = 1 \) on \( E(K) \). Then \( a_0 \in Z(A_c(K)) \).

Proof. In view of the results quoted in the beginning of this section it is sufficient to show that \( a_0 \) is facially continuous.

For a closed set \( B \subset T \), let \( B' = \{ x \in E(K) : a_0(x) \in B \} \). We claim that the closed set \( B' \) is a dilated set. Let \( \mu \) be a maximal probability measure with \( x_0 = r(\mu) \in B' \). Since
\[ 1 = | a_0(x_0) | = \int_{E(K)} a_0 d\mu \leq \int_{E(K)} |a_0| d\mu \leq 1, \]
we get that \( a_0 \equiv a_0(x_0) \) on \( \text{Supp } \mu \) and hence \( \text{Supp } \mu \subset B' \).

It now follows from a result of [2] that \( F, \) the closed convex hull of \( B' \), is a split face and hence \( \{ x \in E(K) : a_0(x) \in B \} = F \cap E(K) \) is a facially closed set.

Remark. When \( K \) is a simplex, \( a \in A_c(K) \) is an extreme point of the closed unit ball of \( A_c(K) \) iff \( | a | = 1 \) on \( E(K) \) iff \( a \in Z(A_c(K)) \) and is an extreme point of the closed unit ball of \( Z(A_c(K)) \).

Corollary 3.3. If \( K \) is a compact Choquet simplex then for any onto isometry \( \Phi \) of \( A_c(K) \), \( \exists \) an affine homeomorphism \( Q \) of \( K \) such that
\[ \Phi(a)(x) = a(Q(x)) \cdot \Phi(1)(x) \quad \forall x \in E(K). \]
Proof. We have observed in the proof of Theorem 3.1 that $|\Phi(1)| = 1$ on $E(K)$, hence the conclusion follows from Corollary 3.2 and Theorem 3.1.

Remark. These results generalize the classical Banach-Stone theorem dealing with the isometries of $C_c(X)$, where $X$ is a compact Hausdorff space; also generalized is the work of A. J. Lazar [3] on isometries of $A(K)$.

4. Example. We end by giving a simple example of a nonsimplicial compact convex set $K$ and an isometry $\Phi$ of $A_c(K)$ which is not of the form described in Theorem 3.1.

Let $K$ be the unit square in $\mathbb{R}^2$ centred at $(0,0)$, so

$$E(K) = \{(x, y): |x| = 1 = |y|\} \cdot K$$

has no proper split faces and hence $Z(A_c(K)) = \{a \cdot 1: a \in \mathbb{C}\}$. Any $f \in A_c(K)$ is of the form $f(x, y) = ax + by + c$ where $a, b, c \in \mathbb{C}$. Define $\Phi(f)(x, y) = cx + by + a$. Now $\|f\| = \max |a \pm b \pm c|$ and $\|\Phi(f)\| = \max |c \pm b \pm a|$ hence $\Phi$ is an isometry. It is obvious that $\Phi$ is onto. But $\Phi(1) = x$, a nonconstant. Hence $\Phi$ is not of the form in Theorem 3.1.

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References


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