ON FOURIER INTEGRAL OPERATORS

A. EL KOHEN

ABSTRACT. We consider operators of the form: \( \int_{-\infty}^{\infty} F_t \varphi(t) \, dt \), where \( F_t \) is a 1-parameter family of Fourier integral operators and \( \varphi(t) \, dt \) a tempered distribution on the real line and show that these operators are sums of pseudo-differential and Fourier integral operators. Here, we give the typical case where \( \varphi(t) \, dt = \text{p.v.}\{1/t\} \). An application to singular integrals on variable curves is given.

THEOREM. Let \( F_t \) be a 1-parameter family of local Fourier integral operators such that the kernel of \( F_t \) has an integral representation of the form:

\[
\int_{\mathbb{R}^2} e^{i\phi(x,y,t,\theta)} a(x,y,t,\theta) \, d\theta
\]

with \( a \) and \( \phi \) depending smoothly on \( t \), a (resp. \( \phi \)) a symbol in the class \( S_{0,0}^\alpha \) i.e.: \( |\partial_\theta^\alpha \phi \partial_\theta^\beta | \leq C_{\alpha,\beta,k,\gamma} (1 + |\theta|)^{-\gamma} \) (resp. nondegenerate phase function) and \( a(x,y,0,\theta) = 1 \) (resp. \( \phi(x,y,0,\theta) = (x-y) \cdot \theta \)). Furthermore, we assume that \( \phi \times \tilde{\phi} \neq 0 \); then the operator \( T = \int_{-1}^1 F_t \, dt/t \) can be written in the form \( T = P + F \) where \( P \) is a \( \psi.d.o \). with symbol in \( S_{1/2,1/2}^0 \) and \( F \) a Fourier integral operator with amplitude in \( S_{1,0}^{-1/2} \).

The proof follows from the asymptotic expansion of the function

\[
\omega(x,y,\theta) = B(\theta) \int_{-1}^1 e^{i\phi(x,y,t,\theta)} a(x,y,t,\theta) \, dt/t
\]

where \( B \) is a \( C^\infty \) function on \( \mathbb{R}^2 \), identically zero near the origin and homogeneous of degree zero for all \( |\theta| \geq 1 \). For simplicity, we will not, in the sequel, make mention of \( B \).

We, now, use Taylor expansion to have:

\[
\phi(x,y,t,\theta) = (x-y) \cdot \theta + tv(x,y,\theta) + t^2 w(x,y,t,\theta)/2
\]

where \( v = v(x,y,\theta) = \phi(x,y,0,\theta) \). We will also write \( w = w(x,y,\theta) = \phi(x,y,0,\theta) \).

From the hypothesis, we have \( v_\theta \times w_\theta \neq 0 \); hence using the homogeneity of \( v \) and \( w \), for any \((x_0,y_0,\theta_0)\), there exists a conic neighborhood \( K \times \Gamma \) of \((x_0,y_0,\theta_0)\) on which either \( |v(x,y,\theta)| \geq a|\theta| \) or \( |w(x,y,\theta)| \geq b|\theta| \) where \( a, b \) are some positive constants depending only on \( K \times \Gamma \). We consider separately the two types of cones. We let

\[
\Gamma_1 = \{ \theta \in \mathbb{R}^2 : v(x,y,\theta) \geq a|\theta| \text{ for all } (x,y) \in K \},
\]

\[
\Gamma_2 = \{ \theta \in \mathbb{R}^2 : w(x,y,\theta) \geq b|\theta| \text{ for all } (x,y) \in K \}.
\]

The two other cones can be considered in a similar fashion.
We first consider $\Gamma_2$. 

**Lemma 1.** The symbol 

$$\sigma_1 = \int_{|t| < 1/\sqrt{w}} \exp \left\{ itv + \frac{t^2}{2} w(x, y, t, \theta) \right\} a(x, y, t, \theta) \frac{dt}{t}$$

belongs to $S^0_{1/2,1/2}(K \times \Gamma_2)$. 

**Proof.** We write 

$$\sigma_1 = \int_{|t| \leq 1/\sqrt{w}} \exp \left\{ itv + \frac{t^2}{2} w \right\} \frac{dt}{t} + E_1$$

where 

$$E_1 = \int_{|t| \leq 1/\sqrt{w}} \exp \left\{ itv + \frac{t^2}{2} w \right\} \left( \exp \left\{ i\frac{t^2}{2} (w(x, y, t, \theta) - w) \right\} a(x, y, t, \theta) - 1 \right) \frac{dt}{t}$$

$$= \int_{|t| \leq 1} \exp \left\{ it\frac{v}{\sqrt{w}} \right\} \frac{1}{t} \left( \exp \left\{ i\frac{t^2}{2} (w(t/\sqrt{w})/w - 1) \right\} a \left( \frac{t}{\sqrt{w}} \right) - 1 \right) \exp \left\{ i\frac{t^2}{2} \right\} dt$$

where $w(t) = w(x, y, t, \theta)$ and $a(t) = a(x, y, t, \theta)$. We then have 

1. $e^{itv/\sqrt{w}}$ belongs to $S^0_{1/2,1/2}(K \times \Gamma_2)$ uniformly in $t$, $|t| \leq 1$, 
2. $w(t/\sqrt{w})/w$ belongs to $S^0_{1,0}(K \times \Gamma_2)$ uniformly in $t$, $|t| \leq 1$. Also $a = a(t/\sqrt{w})$ verifies: 

$$|\partial_{x}^\alpha \partial_{y}^\beta \partial_{t}^k \partial_{\theta}^\gamma a| \leq c_{\alpha, \beta, \gamma, \chi}(1 + |\theta|)^{-(|\gamma| + k/2)}$$

and 

$$\left| \frac{1}{t} \left( \exp \left\{ i\frac{t^2}{2} (w(t/\sqrt{w})/w - 1) \right\} a(t/\sqrt{w}) - 1 \right) \right| \leq (\text{const})(1 + |\theta|)^{-1/2}$$

on $K \times \Gamma_2$. Thus $\frac{1}{t}(\exp\{i\frac{t^2}{2} (w(t/\sqrt{w})/w - 1)\}a(t/\sqrt{w}) - 1)$ and therefore $E_1$, verifies the conclusion of Lemma 1.

For $\sigma_1 - E_1$, we have 

$$\sigma_1 - E_1 = \int_{|t| \leq 1} e^{it\frac{v}{\sqrt{w}}} e^{it^2} \frac{dt}{t} = \int_{|t| \leq 1} \exp \left\{ it\frac{v}{\sqrt{w}} \right\} \frac{dt}{t}$$

$$+ \int_{|t| \leq 1} \exp \left\{ it\frac{v}{\sqrt{w}} \right\} \left( \exp \left\{ i\frac{t^2}{2} \right\} - 1 \right) \frac{dt}{t} = \sigma_{11} + \sigma_{12}.$$ 

$\sigma_{12}$ is easily seen to be in $S^0_{1/2,1/2}(K \times \Gamma_2)$ since $v/\sqrt{w}$ is in $S^1_{1/2,1/2}(K \times \Gamma_2)$ and $\frac{1}{t}(e^{it^2} - 1)$ is a smooth function. For $\sigma_{11}$, we write 

$$\sigma_{11} = \int_0^{v/\sqrt{w}} \sin u \frac{du}{u} = F \left( \frac{v}{\sqrt{w}} \right)$$

from which one deduces that $\sigma_{11}$ belongs to $S^0_{1/2,1/2}(K \times \Gamma_2)$. This completes the proof of Lemma 1.
We now wish to estimate,

$$\sigma_2 = \int_{|t| \leq 1} e^{i\phi(x,y,t,\theta)} a(x,y,t,\theta) \frac{dt}{t}.$$ 

We let \( t(x,y,\theta) \) be such that \( \phi(x,y,t(x,y,\theta),\theta) = 0 \). From the hypothesis we have \( \phi(x,y,t(x,y,\theta),\theta) \neq 0 \). Also, \( t(x,y,\theta) \) is clearly homogeneous of degree 0 in \( \theta \).

**Lemma 2.** The operator corresponding to \( \sigma_2 \) is a Fourier integral operator with phase function \( \phi(x,y,\theta) = \phi(x,y,t(x,y,\theta),\theta) \) and amplitude in \( S^{-1/2}_{1,0}(K \times \Gamma_2) \).

**Proof.** We use the stationary phase method to write the full expansion of \( \sigma_2 \) whose first term is

$$\frac{e^{i\pi/4}}{t(x,y,\theta)} e^{i\phi(x,y,t(x,y,\theta),\theta)} a(x,y,t(x,y,\theta,\theta)} \sqrt{\phi(x,y,t(x,y,\theta),\theta)}.$$

Clearly \( \tilde{\phi} = \phi(x,y,t(x,y,\theta),\theta) \) is homogeneous of degree 1 in \( \theta \). Also from \( \phi(x,y,t(x,y,\theta),\theta) = 0 \), we see that

$$\tilde{\phi}_x = \phi_x(x,y,t,\theta)|_{t=t(x,y,\theta)}, \quad \tilde{\phi}_\theta = \phi_\theta(x,y,t,\theta)|_{t=t(x,y,\theta)},$$

$$\tilde{\phi}_{x\theta} = \phi_{x\theta}(x,y,t,\theta)|_{t=t(x,y,\theta)}$$

which shows that \( \tilde{\phi} \) is nondegenerate since \( \phi \) is. Also from \( \phi(x,y,t(x,y,\theta),\theta) = 0 \) we have \( \tilde{\phi}_x + \tilde{\phi}_t = 0 \); hence \( t_x = -\phi/\phi \); in particular \( t_x \) is bounded on \( K \times \Gamma_2 \). Similarly for \( t_y \) and all higher derivatives in \( x \) and \( y \) of \( t(x,y,\theta) \). Now, it is clear that

$$\frac{1}{t(x,y,\theta)} a(x,y,t(x,y,\theta,\theta)} \sqrt{\phi(x,y,t(x,y,\theta),\theta)}$$

belongs to \( S^{-1/2}_{1,0}(K \times \Gamma_2) \). The expansion given by the stationary phase method gives an asymptotic expansion of the amplitude (use formula 2.14, p. 431, [2] with \( \rho = 1 \ f(t) = \phi(x,y,t,\phi) \) and \( g(t) = a(t)/t \) and the fact that \( t(x,y,\theta) \) stays away from 0 on \( K \times \Gamma_2 \)).

We now consider \( \Gamma_1 \).

**Lemma 3.** The symbol

$$\nu_1 = \int_{|t| \leq 1} \frac{a(x,y,t(x,y,\theta,\theta)}{t(x,y,\theta)} \sqrt{\phi(x,y,t(x,y,\theta),\theta)}$$

belongs to \( S^0_{1/2,1/2}(K \times \Gamma_1) \).

**Proof.** We write

$$\nu_1 = \int_{|t| \leq 1} \frac{a(x,y,t(x,y,\theta,\theta)}{t(x,y,\theta)} \sqrt{\phi(x,y,t(x,y,\theta),\theta)}$$

where

$$F_1 = \int_{|t| \leq 1} \exp\left\{ itv + \frac{t^2}{2} w(t) \right\} a(x,y,t,\theta) - 1 \frac{dt}{t}$$

$$= \frac{1}{\sqrt{\theta}} \int_{|t| \leq 1} e^{it\sqrt{\theta}} \frac{1}{t/\sqrt{\theta}} \left( \exp\left\{ \frac{t^2}{2} \left( w(x,y,t,\theta) - w \right) \right\} \right)$$

$$\cdot a\left( x,y,t/\sqrt{\theta},\theta \right) - 1 \right\} e^{it^2/2} dt.$$
which is easily seen to be in $S^{-1/2}_{1,0}(K \times \Gamma_1)$ (see similar argument for $E_1$).

For $\nu_1 - F_1$, we have

$$\nu_1 - F_1 = \int_{|t| \leq 1} \exp \left\{ \frac{t^2 w}{2} \right\} \frac{dt}{t}$$

Both terms are clearly in $S^0_{1/2,1/2}(K \times \Gamma_1)$ which finishes the proof of Lemma 3.

We now let $\psi_0(u)$ be a smooth function, $\psi_0(-u) = \psi_0(u)$, $\psi_0 \equiv 1$ for $-\frac{1}{2} \leq u \leq \frac{1}{2}$ and $\psi_0 \equiv 0$ for $|u| \geq 1$ and put $\psi_1 = 1 - \psi_0$. The conclusion of Lemma 3 is clearly also valid for

$$\nu'_1 = \int_{-1}^{1} \exp \left\{ \frac{itv + \frac{t^2 w}{2}}{-v} \right\} a(x, y, t, \theta) \psi_0(tv) \frac{dt}{t}.$$

**LEMMA 4.** The operator corresponding to

$$\nu'_2 = \int_{-1}^{1} \exp \left\{ \frac{itv + \frac{t^2 w}{2}}{-v} \right\} a(x, y, t, \theta) \psi_1(tv) \frac{dt}{t}$$

is a Fourier integral operator with phase function equal to $\varphi(x, y, \pm 1, \theta)$ and amplitude in $S^{-1/2}_{1,0}(K \times \Gamma_1)$.

**PROOF.** We write

$$\nu'_2 = \int_{-\sqrt{v}}^{\sqrt{v}} e^{it\sqrt{v}} \exp \left\{ \frac{t^2 w}{2} \right\} a(x, y, t, \theta) \psi_1(t) \frac{dt}{t}.$$
We have the following:

**COROLLARY.** Let $\gamma(x,t)$ be a variable curve of class $C^\infty$ in $\mathbb{R}^2$ such that $\gamma(x,0) = 0$ and $\dot{\gamma} \times \ddot{\gamma} \neq 0$ then the operator

$$Hf(x) = \text{p.v.} \int_{-1}^{1} f(x - \gamma(x,t)) \frac{dt}{t}$$

is a bounded operator from $L^2_{\text{comp}}(\mathbb{R}^2)$ to $L^2_{\text{loc}}(\mathbb{R}^2)$ (see [3]).

**PROOF.** We write

$$f(x - \gamma(x,t)) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i[(x-y)\theta - \gamma(x,t)\theta]} f(y) \, dy \, d\theta$$

and apply our result with $a \equiv 1$ and $\phi(x,y,t,\theta) = (x-y)\theta - \gamma(x,t)\theta$. In particular there exists a $\psi.d.o. P$ with symbol in $S^{0}_{1/2,1/2}$ such that $(H - P)^*(H - P)$ is a $\psi.d.o. \text{ with symbol in } S_{1,0}^{-1}$.

**REFERENCES**