SOLUTION OF THE \( \gamma \)-SPACE PROBLEM

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Abstract. This paper disproves the classic conjecture that every \( \gamma \)-space is quasi-metrizable.

A quasi-metric on a set \( X \) is a generalized metric \( d : X \times X \to [0, \infty) \) satisfying the axioms \( d(x, y) = 0 \Leftrightarrow x = y \) and \( d(x, z) \leq d(x, y) + d(y, z) \), but not necessarily the axiom of symmetry [N, W]. As with a metric, the family of all sets \( B_d(x, r) = \{ y : d(x, y) < r \} \), for \( r > 0 \), form a neighbourhood base at each \( x \in X \) for a topology on \( X \). A space \( X \) with such a topology is called quasi-metrizable.

Following [J1], a (an open) neighbournet \( V \) on a space \( X \) is a binary relation on \( X \) such that, for each \( x \in X \), the set \( V[x] \) is a (an open) neighbourhood of \( x \). A neighbournet \( V \) is called a normal neighbournet if there exists a sequence \( \langle V_k : k \in \mathbb{N} \rangle \) of neighbournets with \( V_{k+1}^2 \subseteq V_k \) for each \( k \in \mathbb{N} \) and with \( V_1 \subseteq V \). Clearly, if \( X \) is a quasi-metrizable space with quasi-metric \( d \), and if for some \( n \in \mathbb{N} \) and each \( x \in X \) we have \( B_d(x, 1/n) \subseteq V[x] \), then \( V \) is a normal neighbournet; for we may define the \( V_k \) by \( V_k[x] = B_d(x, 2^{-k}/n) \).

With the above terminology, a \( T_1 \) space \( X \) is called a \( \gamma \)-space if there exists a decreasing sequence \( \langle V_k : k \in \mathbb{N} \rangle \) of neighbournets (called a \( \gamma \)-sequence) such that, for each \( x \in X \), the family \( \{ V_k^2[x] : k \in \mathbb{N} \} \) is a neighbourhood base at \( x \) [LF, J1]. Clearly every quasi-metrizable space is a \( \gamma \)-space since we may define the \( V_k \) by \( V_k[x] = B_d(x, 2^{-k}) \).

The question as to whether every \( \gamma \)-space is quasi-metrizable has been raised frequently (and often independently) in the literature, for example [NC, S, LN, LF, G, J2], and is listed as Classic Problem VIII in [TP]. Indeed, a proof of this question was first claimed in the literature in 1943 [R1, p. 35]; however the argument given was incomplete and the author had later to strengthen his required conditions [R2]. Currently several partial solutions have been obtained [G–K2; J2; B–K1; F2].

In this paper we construct a Hausdorff counterexample to the \( \gamma \)-space conjecture (currently we know of no regular counterexample). If \( U^n \) is a normal neighbournet whenever \( U \) is a neighbournet on \( X \), then the space \( X \) is said to be \( n \)-pretransitive [FL]. Our starting point will be the existence for each \( n \in \mathbb{N} \), as proven in [F1], of a Hausdorff quasi-metrizable space \( X_n \) which is not \( n \)-pretransitive (such a space is the \( (n + 1) \)th power \( M^{n+1} \) of the Michael line \( M \)). Let \( U_n \) be an open neighbournet on
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Xn such that Un n is not a normal neighbournet. If we let X = ∪∞n=1 Xn be the disjoint
topological sum, and define the open neighbournet U on X by U = ∪∞k=1 Un, then
for no k ∈ N is Uk a normal neighbournet. On the other hand each Xn, and hence
X, are quasi-metrizable spaces and thus γ-spaces. From the space X and open
neighbournet U we construct the non-quasi-metrizable γ-space X as follows.

The points of X are the points of X U X2 U X3 U … U X∞. The basic open sets
of X are all sets in Xn of the form \(\{x_1, \ldots, x_{n-1}\}\) \(\times A\) where A is open in X, for
each n ∈ N, together with all sets

\[ \hat{U}_{x,k} = \{x\} \cup \bigcup_{n=k}^{\infty} (\{x_1, \ldots, x_{n-1}\} \times U^{n-k+1}[x_n]) \]

where k ∈ N and x = \(\langle x_1, x_2, \ldots \rangle\) ∈ X∞.

To show that \(\hat{X}\) is a y-space, let \(\langle V_k : k \in N \rangle\) be a y-sequence for X with V1 ⊆ U
(for example, let \(\langle W_k : k \in N \rangle\) be any y-sequence for X and take V1 = W1 \(\cap U\)).
Define neighbournets \(\hat{V}_k\) on \(\hat{X}\) as follows: If x = \(\langle x_1, \ldots, x_n \rangle\) ∈ Xn then \(\hat{V}_k[x] = \langle x_1, \ldots, x_{n-1} \rangle \times V_k[x_n]\), while if x ∈ X∞ then \(\hat{V}_k[x] = \hat{U}_{x,k}\). Then \(\langle \hat{V}_k : k \in N \rangle\) is
a y-sequence for \(\hat{X}\). For if x = \(\langle x_1, \ldots, x_n \rangle\) ∈ Xn then clearly the sets \((\hat{V}_k)^2[x] = \langle x_1, \ldots, x_{n-1} \rangle \times V_k^2[x_n]\), for k ∈ N, form a neighbourhood base at x. Alterna-
tively, if x = \(\langle x_1, x_2, \ldots \rangle\) ∈ X∞ then the sets

\[ (\hat{V}_k)^2[x] = \{x\} \cup \bigcup_{n=k}^{\infty} (\{x_1, \ldots, x_{n-1}\} \times V_k \circ U^{n-k+1}[x_n]) \]

\[ \subseteq \{x\} \cup \bigcup_{n=k}^{\infty} (\{x_1, \ldots, x_{n-1}\} \times U^{n-k+2}[x_n]) \subseteq \hat{V}_{k-1}[x], \]

for k ∈ N, form a neighbourhood base at x.

To show that \(\hat{X}\) is not quasi-metrizable, suppose d is a quasi-metric for \(\hat{X}\) and
define x_n, z_n ∈ X by induction on n ∈ N as follows.

Assume inductively that x_1, …, x_{n-1} have been defined. Since \(\{x_1, \ldots, x_{n-1}\} \times X \subseteq X_n \subseteq \hat{X}\) is canonically homeomorphic to X, we may choose x_n, z_n ∈ X such
that d(\(\{x_1, \ldots, x_{n-1}, x_n\}\), \(\{x_1, \ldots, x_{n-1}, z_n\}\)) < 1/n but z_n ∉ U^n[x_n]; for otherwise
U^n would be a normal neighbournet.

Having completed the induction, let x = \(\langle x_1, x_2, \ldots \rangle\) ∈ X∞. Find m ∈ N such
that B_d(x, 2/m) \(\subseteq \hat{U}_{x,1}\). Choose n ≥ m such that d(x, \(\langle x_1, \ldots, x_n\rangle\)) < 1/m; then
d(\(\langle x_1, \ldots, x_n\rangle, \langle x_1, \ldots, x_{n-1}, z_n\rangle\)) < 1/n ≤ 1/m but d(x, \(\langle x_1, \ldots, x_{n-1}, z_n\rangle\)) ≥ 2/m
since \(\langle x_1, \ldots, x_{n-1}, z_n\rangle \notin \hat{U}_{x,1}\). Thus d does not satisfy the triangle inequality. This
proves that the space X is not quasi-metrizable.

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