BAIRE SECTIONS FOR GROUP HOMOMORPHISMS

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Abstract. The following result is proved: Let X and Y be compact topological groups and p be a continuous group homomorphism from Y onto X. Then there exists a map q from X to Y such that \( p \circ q = \text{id}_X \) and \( q^{-1}(B) \) is a Baire set in Y for every Baire subset B of X.

1. Introduction. As pointed out by Rieffel [7], Baire measurable sections for group homomorphisms can be used to construct certain well-behaved extension groups. This motivated Kupka [4] to ask the following question: Given a locally compact group Y, a closed subgroup H of Y and the canonical map p from Y onto the space Y/H of left cosets of Y, does there exist a Baire measurable map \( \varphi: Y/H \rightarrow Y \) with \( p \circ \varphi = \text{id}_{Y/H} \)? We will show that the answer is “yes” provided Y is compact and H is a normal subgroup.

2. Preliminaries. Let X and Y be compact Hausdorff spaces, \( \mathcal{B}_0(X) \) and \( \mathcal{B}_0(Y) \) their respective Baire \( \sigma \)-fields. A map \( f: X \rightarrow Y \) is called Baire measurable iff \( f^{-1}(B) \in \mathcal{B}_0(X) \) for all \( B \in \mathcal{B}_0(Y) \). A map \( \Phi \) from X to the nonempty subsets of Y is said to be a correspondence from X to Y (correspondences are also called multifunctions or set-valued functions in the literature). By \( G(\Phi) \) we denote the graph \( \{(x, y) \in X \times Y \mid y \in \Phi(x)\} \) of \( \Phi \). \( \Phi \) is called upper semi-continuous (u.s.c.) iff, for every open subset U of Y, the set \( \{x \in X \mid \Phi(x) \subset U\} \) is open in X. A compact-valued correspondence \( \Phi \) is u.s.c. if and only if \( G(\Phi) \) is closed in \( X \times Y \).

A map \( f: X \rightarrow Y \) is called a selection for \( \Phi \) iff \( f(x) \in \Phi(x) \) for all \( x \in X \). A compact Hausdorff space X is said to have the Bockstein separation property (BSP) iff any two disjoint open subsets of X can be separated by open \( \mathcal{F}_\sigma \)-sets (cf. Pelczyński [6, Definition 5.9]). A classical theorem of Bockstein [1] states that an arbitrary product of compact metrizable spaces has the BSP. The same is true for compact topological groups (cf. Pelczyński [6, Theorem 7.5 and Corollary 5.11]).

3. A selection lemma. The following lemma will be used in the proof of our main theorem but may also be of some interest in itself.

Lemma. Let X be a compact Hausdorff space with the BSP, Z a compact metrizable space, and \( \Phi \) an u.s.c. compact-valued correspondence from X to Z. Then \( \Phi \) has a Baire measurable selection.
Proof. We first note that, due to the fact that $X$ has the BSP, the following holds.

\begin{equation}
\text{(\star)} \quad \text{For every subset } F \text{ of } X \text{ the set } \dot{F} \text{ is a Baire set}
\end{equation}

(\text{where } \overline{A} \text{ and } \dot{A} \text{ denote the closure and the interior of a set } A \text{ respectively}). \text{ To show this let } F \text{ be a subset of } X. \text{ BSP implies that there is an open Baire set } B \text{ such that } \dot{F} \subset B \text{ and } B \cap (X \setminus \overline{F}) = 0. \text{ This implies } \dot{F} \subset B \subset \overline{F}, \text{ hence } \dot{F} = B \text{ because } B \text{ is open.}

\text{We will now show that there is a compact-valued correspondence } \Phi \text{ from } X \text{ to } Z \text{ such that}

(i) $\Phi(x)$ is a subset of $\Phi(x)$ for all $x$ in $X$,

(ii) $\{x \in X \mid \Phi(x) \cap A \neq \emptyset \}$ is a Baire subset of $X$ for all closed subsets $A$ of $Z$.

\text{Suppose for the moment that there is such a } \Phi. \text{ Then the selection theorem of Kuratowski and Ryll-Nardzewski [5] implies that } \Phi \text{ has a Baire measurable selection.}

\text{Since such a selection is also a selection for } \Phi \text{ our lemma will follow.}

\text{To construct } \Phi \text{ we define for each } x \text{ in } X \text{ a collection } \mathcal{S}_x \text{ of nonempty subsets of } Z \text{ by}

\begin{equation}
\mathcal{S}_x := \left\{ B \subset Z \mid \text{B open, } x \in \Phi^{-1}(B) \right\}
\end{equation}

where $\Phi^{-1}(B) := \{ x \in X \mid \Phi(x) \subset B \}$. \text{ We claim that } \mathcal{S}_x \text{ has the finite intersection property. This is a consequence of the following facts:}

1. $\mathcal{S}_x(F \cap G) = \mathcal{S}_x(F) \cap \mathcal{S}_x(G)$ for any two subsets $F$ and $G$ of $Z$,

2. $\mathcal{S}_x(U_1 \cap U_2) = \mathcal{S}_x(U_1) \cap \mathcal{S}_x(U_2)$ for any two open subsets $U_1$ and $U_2$ of $X$,

3. $\mathcal{S}_x(B)$ is open for every open subset $B$ of $Z$ because $\Phi$ is u.s.c.

\text{Therefore } \Phi(x) := \bigcap \{ B \mid B \in \mathcal{S}_x \} \text{ defines a compact-valued correspondence } \Phi \text{ from } X \text{ to } Z.

\text{To show that } \Phi \text{ satisfies (i), assume that there are } x \text{ in } X \text{ and } z \text{ in } Z \text{ such that } z \in \Phi(x) \setminus \Phi(x). \text{ Because } Z \text{ is regular there is an open neighborhood } U \text{ of } z \text{ with } U \cap \Phi(x) = \emptyset. \text{ This implies } x \in \Phi^{-1}(Z \setminus U), \text{ hence } Z \setminus U \in \mathcal{S}_x \text{ and therefore } z \in U \cap (Z \setminus U) = U \cap (Z \setminus U) = \emptyset \text{ which is absurd.}

(ii) \text{ is equivalent to}

(ii) $\Phi^{-1}(U)$ is a Baire set for every open subset $U$ of $Z$.

\text{So let } U \subset Z \text{ be open. Since } Z \text{ is metrizable there exists an increasing sequence } (B_n)_{n \in \mathbb{N}} \text{ of open sets such that } \bigcup_n B_n = \bigcup_n \overline{B_n} = U. \text{ We show that}

\begin{equation}
\Phi^{-1}(U) = \bigcup_n \Phi^{-1}(B_n)
\end{equation}

holds, from which (ii)' will follow because each of the sets $\overline{\Phi^{-1}(B_n)}$ is a Baire set by (\star).

For $x \in \Phi^{-1}(B_n)$ we have $B_n \in \mathcal{S}_x$, hence $\Phi(x) \subset \overline{B_n} \subset U$, which proves one of the required inclusions. To prove the other one let $\Phi(x)$ be contained in $U$. This
implies that $B \subseteq U$ holds for some $B \in \mathcal{B}_x$. $B$ being compact there is an $n \in \mathbb{N}$ with $\bar{B} \subseteq B_n$. Therefore $x \in \Phi_n(B) \subseteq \Phi_n(B_n)$ and the selection lemma is proved.

**Remarks.** (1) Note that in the situation of the lemma the inverse image \(\{x \in X\mid \Phi(x) \cap A \neq \emptyset\}\) of a closed set $A \subset Z$ under $\Phi$ need not be Baire measurable. Therefore, the theorem of Kuratowski and Ryll-Nardzewski applied to $\Phi$, in general only yields a Borel measurable selection for $\Phi$.

(2) The lemma, even in a slightly more general form, can also be derived from the main theorem in [2, Theorem 1, p. 343]. The proof given here uses methods similar to those employed in proving that general theorem.

4. **Main results.** In this section we will establish a selection theorem for correspondences whose graphs are groups. The main ingredients of the proof are the selection lemma and the fact that compact groups have the BSP.

**Theorem.** Let $X$ and $Y$ be compact topological groups and $\Phi$ an u.s.c. compact-valued correspondence from $X$ to $Y$ such that $G(\Phi)$ is a subgroup of the product group $X \times Y$. Then $\Phi$ has a Baire measurable selection.

**Proof.** (a) First we consider the case $Y = \prod_{i \in I} Y_i$, where each $Y_i$ is a compact metrizable group. For $J \subset I$ let $Y_J = \prod_{i \in J} Y_i$ and $\sigma_j: Y \rightarrow Y_J$, $\sigma_j: X \times Y \rightarrow X \times Y_J$ be the canonical projections. Let $\Phi_j$ be the correspondence from $X$ to $Y_J$ defined by $\Phi_j(x) = \sigma_j(\Phi(x))$. Then we have $G(\Phi_j) = \sigma_j(G(\Phi))$, hence $G(\Phi_j)$ is a compact subgroup of $X \times Y_J$ because $\sigma_j$ is a continuous group homomorphism. In particular, $G(\Phi_j)$ has the BSP. Now let $\Gamma = \{(J, \varphi) \mid J \subset I, J \neq \emptyset, \varphi: X \rightarrow Y_J \text{ Baire measurable selection of } \Phi_j\}$.

We introduce a partial order $\leq$ on $\Gamma$ by $$(J, \varphi) \leq (K, \psi) \iff J \subset K \text{ and } \sigma_j \circ \varphi = \sigma_j \circ \psi \text{ for all } j \in J$$ and claim that $\Gamma$ is nonempty and inductively ordered by $\leq$. For $i \in I$ the correspondence $\Phi_i$ is u.s.c. and takes compact values in the compact metrizable space $Y_i$. Hence, by the selection lemma, $\Phi_i$ admits a Baire measurable selection $\varphi_i$, i.e. $(\{i\}, \varphi_i) \in \Gamma$. Now let $(J, \psi) \in \Gamma$. Let $J = \cup J_\lambda$ and define $\varphi: X \rightarrow Y_J$ by $\varphi(x) = \psi(x)$, if $j \in J_\lambda$.

Then $\varphi$ is a well-defined map. The definition of $\varphi$ and the Baire measurability of the $\varphi_j$'s implies that for each $j \in J$ the map $\sigma_j \circ \varphi$ is Baire measurable. Since the Baire $\sigma$-algebra on $Y_J$ is the smallest $\sigma$-algebra rendering all the maps $\sigma_j$ measurable, it follows that $\varphi$ is Baire measurable. Therefore $(J, \varphi)$ is an upper bound of $(J_\lambda, \psi_\lambda)_{\lambda \in \Lambda}$ in $\Gamma$. By Zorn's lemma there exists a maximal element $(M, \mu)$ in $\Gamma$. To complete the proof of (a) it remains to show $M = I$. Assume the contrary. Then there is a $j \in I \setminus M$. Define a correspondence $\Psi$ from $G(\Phi)$ to $Y_j$ by $\Psi((x, y)) = \{z \in Y_j \mid (y, z) \in G(\Phi)_{M \cup \{j\}}(x)\}$.

The graph of $\Psi$ is equal to $G(\Phi_{M \cup \{j\}})$, hence compact. This implies that $\Psi$ is u.s.c. and compact-valued. Since $G(\Phi_M)$ has the BSP, the selection lemma yields a Baire measurable selection $\psi$ for $\Psi$. Define $\varphi: X \rightarrow Y_{M \cup \{j\}}$ by $\varphi(x) = (\mu(x), \psi(x, \mu(x)))$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Then $\varphi$ is obviously a selection for $\Phi_{\mu(j)}$. To show that $\varphi$ is Baire measurable we have to check the measurability of the maps $\pi_i \circ \varphi$ with $i \in M \cup \{j\}$. For $i \in M$ it follows from $\pi_i \circ \varphi = \pi_i \circ \mu$. Moreover, we have $\pi_j \varphi(x) = \psi(x, \mu(x))$ for all $x \in X$. Since $x \mapsto (x, \mu(x))$ is Baire measurable as a map into $X \times Y_M$, taking values in $G(\Phi_M)$, it is also Baire measurable as a map into $G(\Phi_M)$ because $G(\Phi_M)$ is compact. Hence $\pi_j \circ \varphi$ is Baire measurable as a composition of Baire measurable maps. Thus $(M \cup \{j\}, \varphi)$ is an element of $\Gamma$ strictly larger than the maximal element $(M, \mu)$, a contradiction.

(b) To prove the general case we observe that every compact topological group $Y$ is a subgroup of a product $\Pi Y_i$ of compact metrizable groups $Y_i$, because it is a projective limit of such groups (cf. e.g. Higgins [3, p. 98, Theorem A"']). Hence by (a) there exists a selection $\varphi$ of $\Phi$ which is Baire measurable as a map into $\Pi Y$. As before we see that it is also Baire measurable as a map into $Y$. Hence the theorem follows.

Important examples of correspondences satisfying the assumptions of our theorem are given by $\Phi = p^{-1}$ where $p$ is a continuous homomorphism from one compact group onto another. This immediately leads to the following corollary.

**Corollary.** Let $X$ and $Y$ be compact topological groups and $p: Y \to X$ a continuous surjective homomorphism. Then there exists a Baire measurable map $\varphi: X \to Y$ with $p \circ \varphi = \text{id}_X$.

In particular the result announced in the introduction holds.

**Remark.** The map $\varphi$ in the corollary can be chosen in such a way that it maps the identity element onto the identity element (define a new section by $x \mapsto \varphi(e)^{-1}\varphi(x)$). Therefore one always has measurable cross sections in the sense of Rieffel [7, p. 872], provided the groups involved are compact.

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**Added in proof.** Using the same methods, it can be shown that the answer to Kupka's question—mentioned in the introduction—remains "yes" even if the normality condition on the subgroup $H$ is dropped.

**References**


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