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HOMOTOPY IDEMPOTENTS ON FINITE-DIMENSIONAL COMPLEXES SPLIT[1]

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ABSTRACT. We prove that (unpointed) homotopy idempotents on finite-dimensional complexes split, and describe some geometric consequences.

1. Introduction. A *homotopy idempotent* on a space X is a map $f\colon X \to X$ such that $f^2 \simeq f$. It is said to *split* if there are maps $W \xrightarrow{u} X \xrightarrow{v} W$ with $vu \simeq 1_W$, and $uv \simeq f$. It is well known (see E. M. Brown [2], D. A. Edwards and R. Geoghegan [8], P. Freyd [10]) that *pointed* homotopy idempotents on *pointed connected* CW complexes split. Questions about splitting unpointed homotopy idempotents have arisen in several areas. In homotopy theory (Freyd and Heller [11], Heller [13]) this question is closely linked with Brown's representation theorem for half-exact functors. This question is also closely related with the study of FANR's (fundamental absolute neighborhood retracts). A compact metric space C is a FANR if it is a shape (fundamental) retract of a (compact) ANR (absolute neighborhood retract) A, that is, if there is a map j: $C \to A$ and a shape map $r\colon A \to C$ with rj shape equivalent to 1_C. By J. West [16], A is homotopy equivalent to a finite complex X. Then, cf. [8], the composite mapping jr induces a homotopy idempotent on X. In pro-homotopy (D. A. Edwards and Hastings [9]), the question of splitting homotopy idempotents is a special case of the more general question of whether weak pro-homotopy equivalences are strong pro-homotopy equivalences. Similar questions arise in shape theory (J. Dydak and Hastings [6], Dydak and J. Segal [7], Edwards and R. Geoghegan [8]: Is every shape equivalence a strong shape equivalence?) and proper homotopy theory (T. A. Chapman [3], Chapman and L. Siebenmann [4, Appendix II], and Edwards and Geoghegan [8]: Is every weak proper homotopy equivalence a proper homotopy equivalence?).

Recently, Dydak and P. Minc [5], and Freyd and Heller [11] independently found an unpointed homotopy idempotent on an infinite-dimensional complex which does not split. See §2. This answered the pro-homotopy question in the negative. However, the shape and proper homotopy questions remained open, because they involve implicit finiteness restrictions.

We shall prove the following.

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THEOREM 1. *(Unpointed) homotopy idempotents on finite-dimensional complexes split.*

COROLLARY. *Every FANR is a pointed FANR.*

We need consider, without loss of generality, pointed, connected CW complexes. All maps, with such obvious exceptions as deck-transformations of covering spaces, preserve basepoints. We shall consider both pointed ($\sim$) and unpointed ($\simeq$) homotopies.

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2. Proof of Theorem 1. We begin by recalling an earlier result [5, 11]. Let $f\colon X \to X$ be a homotopy idempotent, and let $H\colon f^2 \simeq f$ be a homotopy. Let F be the group with presentation

$$\left\langle x_0, x_1, \ldots \mid x_i x_j x_i^{-1} = x_{j+1}, \text{all } i < j \right\rangle.$$

Then F admits an endomorphism ϕ with $\phi x_i = x_{i+1}$ for all i. Under the homotopy H, the basepoint of X traces out an element ξ_0 in $\pi_1 X$. Then there is a unique homomorphism $\Phi\colon F \to \pi_1 X$ such that $\Phi x_0 = \xi_0$ and $\Phi\phi = (\pi_1 f)\Phi$.

THEOREM 2 [5, 11]. *The homotopy idempotent $f\colon X \to X$ fails to split if and only if Φ is injective. In particular, the Eilenberg-Mac Lane space $K(F, 1)$ has an unsplit homotopy idempotent, the induced map $K(\phi, 1)$.*

If a homotopy idempotent f splits, then f has an image. (Of course, images do not exist, in general in homotopy theory.) Here is a candidate for homotopy "image" of f. Let Y be the homotopy of the diagram colimit of the diagram

$$X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \cdots. \tag{2.1}$$

For an explicit construction, let Y be the Milnor telescope [13], of (2.1), that is,

$$Y = \mathrm{Tel}\left(X \xrightarrow{f} X \xrightarrow{f} X \to \cdots \right) = M_f \cup_X M_f \cup_X M_f \cup_X \cdots. \tag{2.2}$$

Let $i_k\colon X \to Y$ include X as the base of the kth mapping cylinder in Y. Then

(2.3)
(a) $i_{k+1} f \sim i_k$,
(b) $g i_{k+1} \sim i_k$, and
(c) $\pi_* Y = \mathrm{colim}\left(\pi_* X \xrightarrow{f_*} \pi_* X \xrightarrow{f_*} \cdots \right)$,
(d) $H_* Y = \mathrm{colim}\left(H_* X \xrightarrow{f_*} H_* X \xrightarrow{f_*} \cdots \right)$.

Thus for all q, $H_q Y = \bigcup_k \mathrm{Im}(H_q i_k)$. By (2.3)(b), all of the images $\mathrm{Im}(H_q i_k)$ are isomorphic. Thus unless $H_q Y = 0$, all $\mathrm{Im}(H_q i_k) \neq 0$.

Similarly form

$$\tilde{Y} = \mathrm{Tel}\left(\tilde{X} \xrightarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{f}} \cdots \right). \tag{2.4}$$

Then, up to homotopy, $\tilde{Y}$ is the universal cover of Y (compare $\pi_* Y$ and $\pi_* \tilde{Y}$) and analogues of (2.3) hold for $\tilde{Y}$. We consider the left action of fundamental groups on universal covering spaces and their homology. Then for any element ξ of $\pi_1 X$, the following equivariance conditions hold:

$$\tilde{f}\xi \simeq ((\pi_1 f)\xi)\tilde{f}; \quad \tilde{i}_k \xi \simeq ((\pi_1 \tilde{i}_k)\xi)\tilde{i}_k, \qquad k = 0, 1, \ldots. \tag{2.5}$$

We may now proceed to the proof of the theorem. Let $f\colon X \to X$ be a homotopy idempotent on a finite-dimensional complex. Assume that f does not split. First observe that by taking a suitable covering space we may assume that $\pi_1 X = F$, $\Phi = 1_F$, and $\phi = \pi_1 f$. We may then compute, the homotopy "image" of f in (2.2) and (2.4):

$$\begin{gathered} \pi_1 Y = G = \left\langle \ldots, x_{-1}, x_0, x_1, \ldots \mid x_i x_j x_i^{-1} = x_{j+i}, \text{all } i < j \right\rangle \quad \text{and} \\ (\pi_1 i_k) x_j = x_{j-k}, \qquad j = 0, 1, \ldots. \end{gathered} \tag{2.6}$$

For the deck-transformations on the universal coverings, (2.6) yields the formula $\tilde{i}_k x_j \simeq x_{j-k} \tilde{i}_k$.

The homotopy $\tilde{f}^2 \simeq \tilde{f}$ lifts to a homotopy $\tilde{f}^2 \simeq x_0 \tilde{f}$. Thus for all $k > 0$, $\tilde{f}^{k+1} \simeq x_0 f^k$ and, composing with i_{k+1}, $\tilde{i}_0 \simeq x_{-k-1} \tilde{i}_1$. Thus all x_l, $l \leqslant -2$, operate in the same way on the subgroup $\mathrm{Im}(H_q \tilde{i}_1) \subset H_q Y$. For each integer $n > 0$, let T_n be the subgroup of G generated by the n elements

$$x_{-3}^{-1} x_{-2}, x_{-6}^{-1} x_{-5}, \ldots, x_{-3n}^{-1} x_{-3n+1}. \tag{2.7}$$

It is shown in **[11**, cf. also **6, 7]** that T_n is free abelian on these n generators; we have just seen that it operates trivially on $\mathrm{Im}(H_q i_1)$.

Because X is finite-dimensional, there is a largest degree r such that $H_r \tilde{Y} \neq 0$. Also, for each n, let Y_n be the covering space of Y with fundamental group T_n, and compute $H^* Y_n$ by the spectral sequence of the covering $\tilde{Y} \to Y_n$; cf. **[14]** for a detailed description. In this spectral sequence $E^2_{pq} = H_p(T_n, H_q Y)$. Thus $E^2_{pq} \neq 0$ if $p > n$ or $q > r$. Accordingly

$$E^2_{nr} = E_{nr} = H_n(T_n, H_r Y) = (H_r Y)^{T_n}, \tag{2.8}$$

where $(H_r Y)^{T_n}$ denotes the subgroup fixed under T_n. The last equality may be obtained directly (cf. **[14]**) or using Poincaré duality in local coefficients (following a suggestion of the referee):

$$H_n(K(T_n, 1), H_r Y) \cong H^0(K(T_n, 1), H_r Y). \tag{2.9}$$

Thus

$$H_{n+r} Y_n = (H_r Y)^{T_n} = \mathrm{Im}(H_r \tilde{i}_1) \neq 0, \qquad n = 1, 2, \ldots. \tag{2.10}$$

Hence the dimension of Y, and hence also of X, is infinite, a contradiction. This completes the proof. □

3. A geometric application. We sketch a proof of the following, as an example of the use of Theorem 1 in geometric topology.

THEOREM 3. *Every (compact) FANR is a pointed FANR.*

PROOF. Let C be a (compact) FANR. Following **[8]**, as explained in our Introduction, C is a shape retract a finite complex X via shape maps j: $C \to X$ and r: $X \to C$. These yield a homotopy idempotent f: $X \to X$. By **[8]**, f splits in weak pro-homotopy through the inverse sequence

$$X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \cdots, \tag{3.1}$$

which defines the shape of C. Also, C has the shape of a complex if and only if f splits in the homotopy category of complexes. (In this case, f splits through the homotopy limit, holim, of (3.1); see A. K. Bousfield and D. M. Kan **[1]**, also **[8]**, **[9]**.) We provide a splitting in the unpointed case; Edwards and Geoghegan **[8]** had only been able to consider pointed shape and pointed FANR's. Our Theorem 1 extends the Edwards-Geoghegan argument to unpointed shape, thus C has the shape of a complex Y. Hence C has the strong shape of Y, and the pointed shape of Y **[9**, §5**]**. The conclusion now follows from **[8]**. □

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