A SIMPLE PROOF OF RADÓ’S THEOREM

ALAN McCONNELL

In this note we show that Radó’s theorem [3] (sometimes called the Radó-Behnke-Stein-Cartan theorem—see [2] and the literature cited there) is an easy consequence of a basic lemma in the analytic theory of Riemann surfaces, often called Weyl’s lemma [1].

Radó’s theorem reads: If $f(z)$ is a complex-valued function continuous on the unit disc $D$ and analytic where it is not zero, then it is analytic in all of $D$.

Weyl’s lemma states: If $\phi$ is a real-valued Lebesgue measurable function on $D$ such that $\int \int_D \phi \cdot \Delta \mu \, dx \, dy = 0$ for every $C^\infty$ function $\mu$ with compact support contained in $D$, then $\phi$ is almost everywhere equal to a harmonic function on $D$.

Here is how to deduce Radó’s theorem from Weyl’s lemma. Let $f(z) = u(z) + iv(z)$; we shall use Weyl’s lemma to show that $u$ and $v$ are harmonic on all of $D$. Let $D = Z \cup N$ where $Z = \{ z \in D : u(z) = 0 \}$ and $N$ is the open subset of $D$ where $u(z)$ is nonzero and, hence, is harmonic. Let $\{ U_i \}$ be a countable locally finite open covering of $N$ by disc-like sets, and let $\{ \epsilon_i(z) \}$ be an associated $C^\infty$ partition of unity. Finally, let $\mu(z)$ be a $C^\infty$ “test function” as above. We have

$$\int \int_D u \cdot \Delta \mu \, dx \, dy = \int \int_Z u \cdot \Delta \mu \, dx \, dy + \int \int_N u \Delta \mu \, dx \, dy$$

$$= \int \int_N u \Delta \mu \, dx \, dy$$

since the integral over $Z$, where $u(z) = 0$, is zero. But

$$\int \int_N u \cdot \Delta \mu \, dx \, dy = \sum \int \int_N u \cdot \Delta(\epsilon_i \mu) \, dx \, dy,$$
and
\[
\iint_{U_i} u \cdot \Delta(e^{i\mu}) \, dx\, dy = \iint_{U_i} u \cdot \Delta(e^{i\mu}) \, dx\, dy
\]
\[
= \iint_{U_i} \Delta u \cdot e^{i\mu} \, dx\, dy = 0.
\]
(Note that \(e^{i\mu}\) and all its derivatives are zero on \(\partial U_i\).) Thus \(u\) is harmonic on \(D\), and similarly \(v\) is harmonic on \(D\) also.

To finish the proof of Radó's theorem, we must show that \(u\) and \(v\) are conjugate harmonic functions on all of \(D\). Since \(u\) and \(v\) are obviously harmonic conjugates off their common zero set, it is automatic by continuity that they are harmonic conjugates throughout \(D\). Thus \(f(z) = u(z) + iv(z)\) is analytic on \(D\). Q.E.D.

REFERENCES