**Lo-valued Vector Measures Are Bounded**

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**Abstract.** Every vector measure taking values in $L_0(0,1)$ has bounded range.

The question of whether every vector measure taking values in the space $L_0(0,1)$ is bounded was first raised by Turpin [17]. Turpin showed the existence of an unbounded vector measure with range in a certain nonlocally convex $F$-space. Shortly afterwards, Fischer and Scholer [3, 4] and Labuda [9] demonstrated that a vector measure taking values in an Orlicz space $L_φ$ with $φ$ unbounded will be necessarily bounded. The purpose of this note is to show every $L_0$-valued measure is bounded. This result has applications to stochastic integrals [1, 13, 14, 18].

We shall denote by $I$ the unit interval $(0,1)$ and $B$ is the family of Borel subsets of $I$. $λ$ will denote Lebesgue measure on $B$. The space $L_0 = L_0(I; B, λ)$ consists of all real Borel functions on $I$ with functions agreeing almost everywhere identified. This space is equipped with convergence in measure, which is $F$-normed by

$$||f|| = \int_0^1 \frac{|f(t)|}{1 + |f(t)|} \, dλ(t).$$

A base of neighborhoods for 0 is given by sets of the form $V(ε, M) = \{f ∈ L_0: λ(|f| > M) < ε\}$. Let $(S, Σ)$ be any measurable space. Then a (continuous) submeasure $ν: Σ → R_+$ is a set-function satisfying

$$ν(A) ≤ ν(A ∪ B) ≤ ν(A) + ν(B), \quad A, B ∈ Σ,$$

$$ν(A_n) ↓ 0, \quad \text{whenever } A_n ↓ 0.$$

It is an unsolved problem (Maharam [10]) whether every continuous submeasure has an equivalent measure, i.e. a measure giving the same null sets. A continuous submeasure $μ$ induces a pseudo-metric $d$ on $Σ$ given by $d(A, B) = μ(ΔAB)$. We say $Σ$ is $μ$-separable if $(Σ, d)$ is separable; if $ν$ is a measure on a $σ$-algebra $Σ'$ then a map $h: Σ → Σ'$ is continuous if it is continuous with respect to the induced pseudo-metrics.

If $X$ is an $F$-space and $φ: Σ → X$ is a vector measure, then a continuous submeasure $μ$ is said to be a control submeasure for $φ$ if it is equivalent to the

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submeasure

\|\phi\|(A) = \sup(\|\phi(B)\| : B \in \Sigma, B \subset A)

for \(A \in \Sigma\). Maharam's problem is equivalent to the problem of whether every vector measure with values in an \(F^\ast\)-space has a control measure (cf. [2, p. 14]).

Some further notation will be required. If \(A \in \Sigma\) (or \(B\)) then \(1_A\) denotes the indicator function of \(A\), i.e.

\[
1_A(s) = \begin{cases} 1, & s \in A, \\ 0, & s \notin A. \end{cases}
\]

If \(\mathcal{G}\) is a partition of a set \(A \in \Sigma\) into sets from \(\Sigma\), then \(\Sigma(\mathcal{G})\) denotes the family of all unions of sets from \(\mathcal{G}\).

Note. Shortly after the preparation of this paper, the authors learned that the same results have been obtained independently and somewhat earlier by M. Talagrand [19]. Talagrand's proof of Theorem 1 is slightly different in character although it has some ideas in common.

**Theorem 1.** Every vector measure taking values in \(L_0\) is bounded.

**Proof.** The proof will be accomplished via several reductions of the problem. We shall start from the assumption that there exists an unbounded vector measure \(\phi: \Sigma \to L_0\) defined on some measurable space \((S, \Sigma)\), and derive a contradiction. The idea of the argument is to show that we can assume certain properties and these eventually lead to a contradiction.

We denote a control submeasure for \(\phi\) by \(\mu: \Sigma \to \mathbb{R}_+\). Our first simplifying assumption is

(A1) \(\Sigma\) is \(\mu\)-separable and has no \(\mu\)-atoms.

Clearly (A1) is justified by the fact that if \(\phi\) is unbounded it is also unbounded on some \(\mu\)-separable sub-\(\sigma\)-algebra; atoms can be discarded.

We shall also define a set function \(\theta: \Sigma \to \mathbb{R}\) by setting \(\theta(A)\) to be the supremum of all \(\alpha \geq 0\) such that if \(M > 0\) there exists \(B \in \Sigma, B \subset A\) with

\[
\lambda\{t : |\phi(B; t)| \geq M\} \geq \alpha.
\]

(Here \(\phi(B; t) = \phi(B)(t)\).) Note that \(\theta(\Sigma) > 0\).

**Lemma 1.** If \(A, B \in \Sigma\) are disjoint then

\[
\theta(A \cup B) \leq \theta(A) + \theta(B).
\]

**Proof.** If \(\alpha < \theta(A \cup B)\) and \(M > 0\) there exists \(C \in \Sigma\) with \(C \subset A \cup B\) and

\[
\lambda\{|\phi(C)| \geq 2M\} \geq \alpha.
\]

Hence

\[
\lambda\{|\phi(A \cap C)| \geq M\} + \lambda\{|\phi(B \cap C)| \geq M\} \geq \alpha.
\]

By letting \(M \to \infty\), we see that \(\theta(A) + \theta(B) \geq \alpha\) and the lemma follows.

**Lemma 2.** Let \(\mathcal{E} \subset \mathcal{B}\) consist of all Borel sets \(E\) such that the set \(\{1_E \cdot \phi(A); A \in \Sigma\}\) is bounded in \(L_0\). Then \(\mathcal{E}\) is a \(\sigma\)-ideal of \(\mathcal{B}\); in particular if \(E_n \in \mathcal{E}\) (\(n \in \mathbb{N}\)) then \(\bigcup E_n \in \mathcal{E}\).
PROOF. If $E_n \in \mathcal{E}$ then there exist $0 < c_n < 2^{-n}$ such that
\[ \|c_n \cdot 1_{E_n} \cdot \phi(A)\| \leq 2^{-n}, \quad A \in \Sigma, \ n \in \mathbb{N}. \]
Thus $\sum_{n=1}^{\infty} c_n \cdot 1_{E_n} \cdot \phi(A)$ converges uniformly to $h \cdot \phi(A)$ where $h = \sum c_n \cdot 1_{E_n}$. It follows easily that $\{h \cdot \phi(A): A \in \Sigma\}$ is also bounded. Finally if $g(t) = h(t)^{-1}$ for $h(t) > 0$ and $g(t) = 0$ otherwise, then $\{gh \cdot \phi(A): A \in \Sigma\}$ is bounded. However $gh = \sum_{n=1}^{\infty} c_n \cdot 1_{E_n}$.

In view of Lemma 2 we can find a set $F \in \mathcal{E}$ of maximal measure and if $E \in \Sigma$ then $\lambda(E \setminus F) = 0$. We call $F$, which is unique up to sets of measure zero, the bounded support of $\phi$, and let $I \setminus F$ be the unbounded support of $\phi$. For each $A \in \Sigma$, let $A^*$ be the unbounded support of the measure $B \rightarrow \phi(A \cap B)$. We observe some simple properties of the map $A \rightarrow A^*(\Sigma \rightarrow B)$.

**Lemma 3.**
(a) $X(A^*) = 0$ if and only if $\{\phi(B): B \subseteq A\}$ is bounded.
(b) $(A \cup B)^* = A^* \cup B^*$ up to sets of $\lambda$-measure zero for $A, B \in \Sigma$.
(c) $\phi(A) \leq \lambda(A^*)$, $A \in \Sigma$.
(d) If $\mu(\Delta AB) = 0$ then $\lambda(A^* \Delta B^*) = 0$, $A, B \in \Sigma$.

The proofs of these statements are almost immediate.

The next lemma is, however, crucial in the development of the proof of the theorem.

**Lemma 4.** Given $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\lambda(A^*) < \epsilon$. Hence, if $A, B \in \Sigma$ and $\mu(\Delta AB) < \delta$ then $\lambda(A^* \Delta B^*) < \epsilon$.

**Proof.** Given $\epsilon > 0$ choose $\delta > 0$ such that $\mu(A) < \delta$ implies $\phi(A) \in V(\epsilon/256, 1)$. Fix any $A \in \Sigma$ with $\mu(A) < \delta$ and let $\mathcal{G} = \{B_1, \ldots, B_n\}$ be any partition of $A$.

Let $f_i = \phi(B_i)$ ($1 \leq i \leq n$) and let $\{g_j: 1 \leq j \leq 2^n\}$ be some ordering of the functions $\sum_{i=1}^{n} a_i f_i$ over all choices of signs $a_i = \pm 1$. We consider the map $T: l_1 \rightarrow L_1$ defined by
\[ T(\xi) = \sum_{i=1}^{2^n} \xi_i g_i \quad \text{for} \ \xi = (\xi_i) \in l_1.\]

The set $K = \{T(\xi): \|\xi\| \leq 1\}$ is exactly the absolutely convex hull of the set $\phi(\Sigma(\mathcal{G}))$.

If $h \in K$ then $h = \sum_{j=1}^{n} c_j f_j$ where $-1 \leq c_j \leq 1$. Now by a lemma of Musial, Wojcynski and Ryll-Nardzewski [15] (essentially the same lemma is originally found in Maurey-Pisier [12]), there is a probability measure $P$ on the set $\Omega = \{-1, +1\}^n$ so that for any $x_1, \ldots, x_n \in \mathbb{R}$
\[ P \left\{ \omega: \left| \sum X_i(\omega) x_i \right| \geq \frac{1}{\delta} \left\| \sum c_i x_i \right\| \right\} \geq \frac{1}{\delta}, \]
where $X_i: \Omega \rightarrow \{-1, +1\}$ is the $i$th coordinate map.

Let $E = \{t: \left| \sum c_i f_i(t) \right| \geq 16\}$. Then for $t \in E$
\[ P \left\{ \omega: \left| \sum X_i f_i(t) \right| \geq 2 \right\} \geq \frac{1}{\delta} \lambda(E).\]

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However for each \( \omega \in \Omega \), \( \sum X_i f_i \in V(\epsilon/128, 2) \) and hence \( \frac{1}{2} \lambda(E) \leq \epsilon/128 \) or \( \lambda(E) \leq \epsilon/16 \). Thus \( h \in V(\epsilon/16, 16) \).

We now apply Nikišin’s theorem [16] to the operator \( T \). By examining the proof given in [5] it can be seen that there is a Borel set \( E \) with \( \lambda(E) \geq 1 - \epsilon \) and

\[
\lambda(\{[T\xi] > \tau \} \cap E) \leq 1024/\epsilon \tau, \quad 0 < \tau < \infty.
\]

(An alternative approach to this step may be obtained from results in a forthcoming paper [6].)

Let \( d_g = 1_E \). Then for \( B \in \Sigma(G) \)

\[
\int_B |\phi(B; t)|^{1/2} dt = \int_B |\phi(B; t)|^{1/2} dt \leq 2048/\epsilon.
\]

Consider \( d_g \in L_{\infty}(0, 1) \) as a net over all partitions of \( A \) ordered by refinement. Then \( \{d_g\} \) has a cluster point \( a \), \( 0 \leq a \leq 1 \), a.e. \( \int a(t) |\phi(B; t)|^{1/2} dt \leq 2048/\epsilon \) for \( B \in \Sigma \) with \( B \subset A \). Now \( \int a(t) dt \geq 1 - \epsilon \) and so, if \( b(t) = a(t)^{-1} \) for \( a(t) > 0 \) and \( b(t) = 0 \) otherwise, \( \cdot \cdot a = 1_F \) where \( \lambda(F) \geq 1 - \epsilon \). The set \( \{1_F \cdot \phi(B); B \in \Sigma, B \subset A\} \) is thus bounded in \( L_0 \) and so \( \int \lambda \) \( F \supset A \), i.e. \( \lambda(A^*) \leq \epsilon \).

We now come to our second reduction of the problem. We can assume

(A2) \( \mu \) is a probability measure on \( \Sigma \).

Justification of (A2). For each partition \( G \) of \( S \), \( G = \{B_1, \ldots, B_n\} \) define \( \{C_i; 1 \leq i \leq n\} \) in \( B \) by \( C_i = B_i \cup \bigcup_{j < i} B_j^* \). Define for \( A \in \Sigma \)

\[
\nu_G(A) = \left\{ \sum \lambda(C_i); B_i \cap A \neq \emptyset \right\}.
\]

Then \( \nu G \) is additive on \( \Sigma(G) \), monotone and \( \nu G(S) = \lambda(S^*) > 0 \). Denote by \( \nu \) any pointwise cluster point of the net \( \{\nu_G\} \) of set functions on \( \Sigma \). Then \( \nu(S) = \lambda(S^*) \), \( \nu \) is additive and monotone and \( \nu(B) \leq \lambda(B^*) \), \( B \in \Sigma \). Hence by Lemma 4, \( \nu \) is \( \mu \)-continuous. It follows that \( \nu \) is countably additive and there is a subset \( A \in \Sigma \) so that \( \nu(A) > 0 \), and if \( B \subset A \) with \( B \in \Sigma \) then \( \nu(B) = 0 \) if and only if \( \mu(B) = 0 \), i.e. \( \nu \) and \( \mu \) are equivalent on \( \Sigma \cap A \).

We now achieve our reduction by replacing \( \phi \) by its restriction to \( A \) and \( \mu \) by \( \nu(A)^{-1} \nu \). The new \( \phi \) is still unbounded since \( \lambda(A^*) \geq \nu(A) > 0 \), and of course assumption (A1) remains in force.

Our third reduction is that we can assume

(A3) \( \lambda(A^* \cap B^*) = 0 \) whenever \( A \cap B = \emptyset \).

The justification of (A3) is partially based on an argument of Kwapień [8].

Justification of (A3). Let \( \{B_{n,k}; 1 \leq k \leq 2^n\} \) be; for each \( n \), a partitioning of \( S \) into sets of \( \mu \)-measure \( 2^{-n} \) so that

\[
B_{n,k} = B_{n+1,2k-1} \cup B_{n+k,2k}, \quad 1 \leq k \leq 2^n, \quad n \in \mathbb{N},
\]

and \( \{B_{n,k}; 1 \leq k \leq 2^n, n \in \mathbb{N}\} \) is \( \mu \)-dense in \( \Sigma \).

For given \( \epsilon > 0 \) there exists \( \delta \) so that \( \mu(A) < \delta \) implies \( \lambda(A^*) < \epsilon \). For each \( n \) let \( m = m(n) = \lceil \delta \cdot 2^n \rceil \).

Let \( \psi_n \in L_0 \) be defined by

\[
\psi_n = \sum_{k=1}^{2^n} \chi_{n,k}, \quad \text{where} \quad \chi_{n,k} = 1_{B_{n,k}^*}.
\]
Then \( \{\psi_n\} \) is monotone increasing in \( L_0 \) and integer-valued.

For any \( m \)-subset \( J \) of \( \{1, 2, \ldots, 2^n\} \),

\[
\int_0^1 \max_{i \in J} \chi_{n,i}(t) \, dt \leq \epsilon
\]

and summing over all such sets,

\[
\int_0^1 \sum_{i \in J} \max_{i \in J} \chi_{n,i}(t) \, dt \leq \left( \frac{2^n}{m} \right) \epsilon,
\]

or

\[
\int_0^1 \left( \frac{2^n}{m} - \frac{2^n - \psi_n(t)}{m} \right) \, dt \leq \left( \frac{2^n}{m} \right) \epsilon.
\]

\[
\left( \frac{2^n - \psi_n(t)}{m} \right) = \left( \frac{2^n}{m} \right) \cdot \frac{2^n - m}{2^n} = \frac{2^n - m - \psi_n(t) + 1}{2^n - \psi_n(t) + 1} \leq \left( \frac{2^n}{m} \right) \left( 1 - \frac{m}{2^n} \right) \psi_n(t) = \left( \frac{2^n}{m} \right) \left( 1 - \frac{\delta}{2} \right) \psi_n(t)
\]

whenever \( 2^n > \delta^{-1} \). Thus

\[
\inf_n \int_0^1 \left( 1 - \frac{\delta}{2} \right) \psi_n(t) \, dt \geq 1 - \epsilon.
\]

Applying this to every \( \epsilon > 0 \) we conclude that \( \sup \psi_n = \psi < \infty \) a.e.

Of course, since \( \phi \) is unbounded, we must have \( \psi > 0 \). Hence there exists \( F_0 \in B \) with \( \lambda(F_0) > 0 \) and \( n \in N \) so that

\[
\psi_n(t) = \psi(t) > 0, \quad t \in F_0.
\]

Now there exists \( k, 1 \leq k \leq 2^n \) with \( \lambda(B_{n,k} \cap F_0) > 0 \). Let \( F = B_{n,k} \cap F_0 \).

Since for \( m > n \), \( \sum_{j=1}^{2^n} \chi_{m,j} = \psi_m = \psi_n \) on \( F \), we must have (for fixed \( m \)),

\[
\sum_{B_{m,j} \subseteq B_{n,k}} \chi_{m,j}(t) = 1, \quad t \in F,
\]

so that the sets \( \{B_{m,j} \cap F : B_{m,j} \subseteq B_{n,k}\} \) intersect only in sets of \( \lambda \)-measure zero.

It follows quickly from the \( \mu \)-\( \lambda \)-continuity of the map \( A \mapsto A^* \) that if \( A_1, A_2 \in \Sigma \) with \( A_1 \cap A_2 = \emptyset \) and \( A_1, A_2 \subseteq B_{n,k} \) then

\[
\lambda(F \cap A_1^* \cap A_2^*) = 0.
\]

Now we achieve our reduction by replacing \( \phi \) by the measure \( \phi' \), restricted to \( B_{n,k} \cap F \), \( \phi'(A) = 1_F \cdot \phi(A), A \in \Sigma \), \( A \subseteq B_{n,k} \cap F \). It is again clear that \( \phi' \) is unbounded and we can obtain (A2) by renormalizing \( \mu \). It is not difficult to see that our procedure replaces (for \( A \subseteq B_{n,k} \), \( A^* \) by \( F \cap A^* \) (up to sets of measure zero) and so (A3) now holds.

Under the assumptions (A1)-(A3) we now prove

**Lemma 5.** Given any \( \epsilon > 0 \), disjoint sets \( A_1 \cdots A_n \in \Sigma \) and \( M > 0 \), there exist \( B_i \subseteq A_i, B_i \in \Sigma \) so that for every subset \( J \) of \( \{1, 2, \ldots, n\} \)

\[
\left| \phi\left( \bigcup_{i \in J} B_i \cup \bigcup_{i \notin J} (A_i \setminus B_i) \right) \right| \geq M
\]

on a set of measure at least \( \sum_{i=1}^n \theta(A_i) - \epsilon \).
PROOF. We may choose a constant $K$ so large that

(i) $\lambda\{\phi(B_i) \geq nK + M\} \geq \theta(A_i) - \epsilon/4n$.

Choose $B_i \subset A_i$, $B_i \in \Sigma$ so that $\lambda(B_i) \leq nK + M$. For $J \subset \{1, 2, \ldots, 2^n\}$, let $C = \bigcup_{i \in J} B_i \cup \bigcup_{i \notin J} (A_i \setminus B_i)$. Then for each $i$, let $E_i = \{t: |\phi(B_i; t)| \geq nK + M, t \in A_i\}$. Then $\lambda(E_i) \geq \theta(A_i) - \epsilon/4n - \epsilon/4n^2 \geq \theta(A_i) - \epsilon/2n$. If $t \in E_i$ and $i \in J$ then

$$|\phi(C; t)| \geq |\phi(B_i; t)| - (n - 1)K \geq M$$

except on a set of measure at most $(n - 1)\epsilon/4n^2 < \epsilon/4n$. (Here we use the fact that the sets $A_i^*$ are almost disjoint and (i)).

If $t \in E_i$ and $i \notin J$ then

$$|\phi(C; t)| \geq |\phi(B_i; t)| - (n - 1)K - |\phi(A_i; t)| \geq M$$

except on a set of measure at most $\epsilon/4n$. Hence $\lambda\{|\phi(C)| \geq M\} \geq \sum_{i=1}^n \theta(A_i) - \epsilon$ as the sets $\{E_i: 1 \leq i \leq n\}$ are almost disjoint.

**Lemma 6.** $\theta$ is a measure on $\Sigma$ which is $\mu$-continuous.

**Remark.** Of course (A1)-(A3) are in force here.

**Proof.** By Lemma 1, $\lambda(A \cup B) < \lambda(A) + \lambda(B)$ and by Lemma 5, $\lambda(A \cup B) \geq \lambda(A) + \lambda(B)$ for disjoint $A, B$. As $\lambda(A) \leq \lambda(A^*)$ and by Lemma 4, $A \mapsto A^*$ is continuous, we must have that $\theta$ is $\mu$-continuous and countably additive.

We now make a further reduction; we may assume

(A4) There is a constant $p$, $0 < p < 1$, so that $\theta(A) = p\mu(A)$, $A \in \Sigma$.

**Justification of (A4).** Since $\theta$ is $\mu$-continuous and nonzero ($\phi$ is unbounded), there is a subset $B \in \Sigma$ so that $\theta(B) > 0$ and $\theta$ and $\mu$ are equivalent on $\Sigma \cap B$. Restrict $\phi$ to $B$ and redefine $\mu(A)$ as $\theta(B)^{-1}\theta(A)$ for $A \in \Sigma \cap B$. Let $p = \theta(B)$ and (A4) will hold. Of course since $\theta(B) > 0$, $\phi$ is still unbounded.

Under assumptions (A1)-(A4) we now prove

**Lemma 7.** Let $\Sigma_0$ be a finite subalgebra of $\Sigma$ and suppose $\epsilon, M > 0$. Then there is a set $C \in \Sigma$ independent of $\Sigma_0$ with $\mu(C) = 1/2$ so that

$$\lambda\{|\phi(C)| \geq M\} \geq p - \epsilon.$$

**Proof.** Let $A_1, \ldots, A_n$ be the atoms of $\Sigma_0$. Choose $N$ sufficiently large so that $\mu(B) \leq n/N$ implies $\phi(B) \in V(\epsilon/2, 1)$. Subdivide each $A_i$ into $N$ disjoint sets $(A_{ij}: 1 \leq j \leq N)$ of $\mu$-measure $\mu(A_i)/N$. Now use Lemma 5 to produce $B_{ij} \subset A_{ij}$ so that for any subset $J$ of $L = \{(i, j): 1 \leq i \leq n, 1 \leq j \leq N\}$,

$$\lambda\left\{\phi \left(\bigcup_{j \in J} B_{ij} \cup \bigcup_{L \setminus J} (A_{ij} \setminus B_{ij})\right) \geq M + 1\right\} \geq p - \epsilon/2.$$  

By appropriate choice of $J$ we may suppose that if $D = \bigcup_{j \in J} B_{ij} \cup \bigcup_{L \setminus J} (A_{ij} \setminus B_{ij})$, then

$$\frac{1}{2}\mu(A_i) \leq \mu(D \cap A_i) \leq \frac{1}{2}\mu(A_i) + N^{-1}$$

for each fixed $i$. Choose $D_i \in \Sigma$, $D_i \subset D \cap A_i$ so that $\mu(D_i) = \frac{1}{2}\mu(A_i)$. Let $C = \bigcup D_i$. Then $\mu(D \setminus C) \leq n/N$, and $\lambda\{|\phi(C)| \geq M\} \geq p - \epsilon$ as required. Clearly $C \cap A_i = D_i$. 


We now are in position for the final step in the theorem. Assumptions (A1)-(A4) remain in force. First we determine \( \delta > 0 \) so that \( \mu(A) < \delta \) implies that \( \phi(A) \in V(p/50, 1) \). Next select an integer \( r \) so that \( (1 - \delta/2)^r \leq 9 / 25 \). Select a further integer \( N \) so that \( 2^N > \delta^{-1} \) and \( N > 2^{r+2}/p \) and a constant \( K, K > 2^{N+2} \).

We select, by induction, a sequence \( \{C_n: 1 \leq n \leq N\} \) of sets in \( \Sigma \) and an increasing sequence of constants \( \{M_n: 1 \leq n \leq N\} \) so that

(i) \( \mu(C_n) = \frac{1}{2}, 1 \leq n \leq N \),
(ii) \( C_n \) is independent of the algebra generated by \( \{C_1, \ldots, C_{n-1}\} \) for \( n \geq 2 \),
(iii) \( \lambda_{\{\phi(C_n)\}} \geq M_n \leq p / 16N \),
(iv) \( \lambda_{\{\phi(C_{n+1})\}} \geq M_n + K \geq \frac{1}{2}p, n \geq 1 \),
(v) \( \lambda_{\{\phi(C_1)\}} \geq K \geq \frac{1}{2}p \).

Clearly Lemma 7 implies we can make such a construction. Set \( M_0 = 0 \) for convenience and

\[
E_n = \{t: |\phi(C_n; t)| > M_{n-1} + K\}, \quad n = 1, 2, \ldots, N.
\]

Then \( \sum_{n=1}^{N} \lambda(E_n) \geq \frac{1}{2}NP \). Hence the set of \( t \) which belongs to at least \( \frac{1}{2}NP \) of the sets \( E_n \) has measure at least \( \frac{1}{4}P \). Now use (iii) as well to produce a set \( F \subset I \) with \( \lambda(F) \geq 3P/16 \) such that if \( t \in F \), then \( t \in E_n \) for at least \( \frac{1}{2}NP \) sets \( E_n \) and \( |\phi(C_n; t)| \leq M_n \) for all \( n, 1 \leq n \leq N \).

Let \( A_1, \ldots, A_{2N} \) be the atoms of the finite algebra generated by \( \{C_1, \ldots, C_N\} \) so that \( \mu(A_i) = 2^{-N} \). Let \( f_i = \phi(A_i) \). Let \( u(t) \) \( (t \in I) \) be the decreasing rearrangement of the finite sequence \( \{|f_1(t)|, |f_2(t)|, \ldots, |f_{2N}(t)|\} \).

For fixed \( t \in F \), let \( i_1, \ldots, i_r \) be chosen to be distinct and so that \( |f_{i_k}(t)| = u_k(t) \), \( 1 \leq k \leq r \). Since \( \frac{1}{2}NP > 2^r \) there are two distinct indices \( m \) and \( n \) such that \( A_{i_k} \subset C_m \) if and only if \( A_{i_k} \subset C_n \) \( (1 \leq k \leq r) \), and \( t \in E_m \cap E_n \). Hence

\[
|\phi(C_n; t) - \phi(C_m; t)| \leq \sum_{k=r+1}^{2N} u_k(t) \leq 2^N u_r(t).
\]

However, if \( n > m \), \( |\phi(C_n; t)| \geq M_n + K \) and \( |\phi(C_m; t)| \leq M_m \) so that we conclude

\[
u_r(t) \geq K/2^N \geq 4, \quad t \in F.
\]

Now choose \( q \in N \) so that \( \frac{1}{2} \delta \leq q \cdot 2^{-N} \leq \delta \); this is possible since \( 2^N > \delta^{-1} \). We introduce two sets of random variables \( \{X_1, \ldots, X_{2N}\}, \{Y_1, \ldots, Y_{2N}\} \) defined on some (finite) probability space \( \Omega \). The joint distribution of \( \{X_i: i \leq 2N\} \) is such that a \( q \)-subset of \( \{1, 2, \ldots, 2N\} \) is chosen at random and \( X_i = 0 \) or \( 1 \) according as \( i \) belongs to this subset or \( i \) fails to belong to the subset. \( \{Y_1, \ldots, Y_{2N}\} \) are mutually independent and independent of \( \{X_1, \ldots, X_{2N}\} \) with \( P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2} \).

For any \( \omega \in \Omega \), \( \sum_{i=1}^{2N} X_i(\omega)Y_i(\omega) \phi(A_i) \in V(p/25, 2) \). For fixed \( t \in (0, 1) \), suppose as above \( i_1, \ldots, i_r \) are distinct indices so that \( u_k(t) = |f_{i_k}(t)| \), \( 1 \leq k \leq r \). Let \( \Omega_k \) \( (1 \leq k \leq r) \) be the event that \( X_{i_1} = \cdots = X_{i_{k-1}} = 0 \) but \( X_{i_k} = 1 \). Then by symmetry \( P(\omega \in \Omega_k: \sum X_iY_i f_i(t) \geq u_k(t)) \geq \frac{1}{2}P(\Omega_k) \). Hence

\[
P\left( \sum X_iY_if_i(t) \geq u_r(t) \right) \geq \frac{1}{2}P\left( \bigcup_{k=1}^r \Omega_k \right) \geq \frac{1}{2} \left( 1 - \left( 1 - \frac{q}{2^N} \right)^r \right) \geq \frac{1}{2} \left( 1 - \left( 1 - \frac{\delta}{2} \right)^r \right) > \frac{8}{25}.
\]
Now \( P \otimes \lambda \{ (\omega, t) : \sum X_i Y_i f_i \geq 2 \} \leq p/25 \) and hence \( \lambda \{ t : u_r(t) \geq 2 \} \leq p/8. \) Thus \( \lambda(F) \leq p/8. \) However we originally showed \( \lambda(F) \geq 3p/16 \) so that we have arrived at the desired contradiction and the proof of the theorem is complete.

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