MIDDLE NUCLEUS = CENTER
IN SEMIPRIME JORDAN ALGEBRAS

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ABSTRACT. A. A. Albert showed that the middle nucleus and center coincide for a simple Jordan algebra finite-dimensional over a field of characteristic \( \neq 2 \).
E. Kleinfeld extended this to arbitrary simple Jordan algebras of characteristic \( \neq 2 \). Recently this result has played a crucial role in the structure theory of E. Zelmanov. In this note we extend the result to linear Jordan algebras with no derivation-invariant trivial ideals.

The left, middle, and right nucleus \( N_l(A), N_m(A), N_r(A) \) of any nonassociative algebra \( A \) consists of the elements \( n \in A \) with \( [n, A, A] = 0, [A, n, A] = 0, [A, A, n] = 0 \) respectively, where the associator is given by \( [x, y, z] = (xy)z - x(yz) \).
The nucleus \( N(A) \) consists of the elements in all three nuclei simultaneously, and the center \( C(A) \) consists of the nuclear elements which commute with \( A, [n, A] = 0 \).

A derivation of \( A \) is a linear transformation \( D \) satisfying
\[
D(xy) = D(x)y + xD(y)
\]
and hence necessarily
\[
D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]
\]
as well. From this we see each nucleus is derivation-invariant, \( D(N_l(A)) \subset N_l(A) \) for all \( D \) (the same is true of the center). When \( A \) is commutative we have
\[
[x, y, z] = -[y, z, x],
\]
\[
[x, y, z] + [y, z, x] + [z, x, y] = 0,
\]
so \( C(A) = N(A) = N_l(A) = N_r(A) \subset N_m(A) \). Our goal is to show conversely that for semiprime Jordan algebras the reverse inclusion holds as well. This has become important in Zelmanov’s work [4], where expressions \( [x_3, [x_1, x_2]^2, x_4] \) play a crucial role and one wants to know that if these vanish for all \( x_3, x_4 \) (i.e. \( [x_1, x_2]^2 \) is middle-nuclear) then \( [x_1, x_2]^2 \) is actually central.

We restrain our quadratic sympathies and work entirely with (nonunital) linear Jordan algebras \( J \) over a ring of scalars \( \Phi \) containing \( \frac{1}{2} \). Thus \( J \) has product \( xy \) satisfying the Jordan axioms
\[
xy = yx, \ i.e. \ [x, y] = 0,
\]
\[
(x^2y)x = x^2(yx), \ i.e. \ [x^2, y, x] = 0.
\]
In addition to the left multiplication operator \( L_x(z) = xz \), the \( U \)-operator
\[
U_x(z) = (2L_x^2 - L_{x^2})(z)
\]
plays an important role even in the linear theory. It satisfies the identity

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The inner derivations $D_{x,y}$ are defined by

$$
(9) \quad D_{x,y}(z) = [L_x, L_y](z) = -[x, z, y];
$$

any derivation $D$ satisfies

$$
(10) \quad D(U_{xy}) = UD_x(x) + UD_y(y), \quad [D, D_{x,y}] = DD_{x,y} + D_{x,y}.
$$

A Jordan algebra is semiprime if it contains no trivial ideals $B (UBB = 0)$ and is $D$-semiprime if it contains no nonzero trivial ideals $B$ invariant under all derivations.

**Theorem.** The middle nucleus and center coincide, $N_m(J) = C(J)$, for any $D$-semiprime linear Jordan algebra $J$.

**Proof.** We must show $n \in N_m(J) \Rightarrow n \in N_l(J)$, i.e. $[J, n, J] = 0 \Rightarrow [n, J, J] = 0$. Following Albert [1] we introduce an ideal

$$
(11) \quad B = [n, J, J]
$$

which measures the failure of $n$ to lie in $N_l(J)$. Here $B$ is spanned by all associators $[n, x, y]$ for $x, y \in J$, where

$$
(12) \quad [n, x, y] = [n, y, x]
$$

since $0 = [n, x, y] + [x, y, n] + [y, n, x] = [n, x, y] - [n, y, x] + 0$ by (4), (3), and the definition of $n \in N_m(J)$. Thus $[n, x, y] = \frac{1}{2}([n, x, y] + [n, y, x])$ results from linearizing $[n, x, z]$, so

$$
(13) \quad B \text{ is spanned by all } [n, x, x] \text{ for } x \in J.
$$

To see $B$ is actually an ideal, note $JB$ is spanned by all $y[n, x, x] = -x[n, y, x] + [n, xy, x]$ (by (1), (9)) = $-x[n, y, x] + [n, x, y, x]$ (by (12)) = $-\frac{1}{2}[n, x^2, y] + [n, xy, x]$ (by (1)) $\subseteq [n, J, J] = B$.

Now by (13), $[n, B, J]$ is spanned by elements $[n, [n, x, x], y] = -[[n, x, x], y, n] - [y, n, [n, x, x]]$ (by (4)) = $D_n D_{x,n} (y) + 0$ (by (9)) = $[[D_n, D_{x,n}] - D_{D_{x,n}} (n, x)] (y)$ (by (10)) = 0 (since $D_{x,n}(n) = 0$), so

$$
(14) \quad [n, B, J] = 0.
$$

Then $0 = [n, B, J] \supset [n, U_B J, J] = U_B [n, B, J] + U_B [n, J, J]$ (by (9), (10)) = $0 + U_B B$ implies $U_B B = 0$. If $J$ is semiprime then $B = [n, J, J] = 0$ by (11) and $n \in N_l(J)$.

If $J$ is merely $D$-semiprime we must work a little harder. We claim $C = [N_m(J), J, J] = \sum [n, J, J]$ is a trivial $D$-invariant ideal. $C$ is an ideal since each $[n, J, J]$ is by (11), and it is $D$-invariant by (2) and the fact that $N_m(J)$ is $D$-invariant. As in (14) we have

$$
(14') \quad [n, C, J] = 0
$$

for any $n \in N_m(J)$ since by (12) this is spanned by all $[n, [n', x, x], y] = [[n, n', y], x, x] + [n', [n, x, y], x] + [n', x, [n, x, y]]$ (by (9), (2)) = $0 + 2[n', [n, x, y], x]$ (by (12) and $n' \in N_m(J)) = 2(n'[n, x, x]) - 2n'([n, x, y], x)$ (by (12)) = $[n, x^2, y] - [n, n'y, x^2]$ (by (1), (9)) = 0 (by (12)). Then $0 = [n, C, J] \supset [n, C, J] = [n, C, J]J + C[n, J, J]$ (by (1)) = $0 + C[n, J, J]$ (by (14')); this holds for all $n \in N_m(J)$, so $0 = CC$ and $C$ is trivial as well as $D$-invariant. If $J$ is $D$-semiprime this forces $C = 0$, so $[N_m(J), J, J] = 0$ and $N_m(J) \subset N_l(J)$. $\square$
It is easy to give examples to show that the semiprimeness hypothesis is needed. If we take $J = F/K$ for $F$ the free Jordan algebra on $x, y$, $K$ the ideal generated by all $[a, y, b]$ for $a, b \in F$, then $n = y + K$ is middle-nuclear in $J$ by construction but is not central: $[n, J, J] \neq 0$ since $[y, x, x] \notin K$ ($K$ is graded, vanishing in total degrees $< 3$, and in degree $x^2y$ it is spanned by $[x, y, x] = 0$, so $[y, x, x] \notin K$).

We can also give special examples $J = A^+$ for $A$ an associative algebra. Here $n \in N_m(J) \leftrightarrow [[J, J], n] = 0$ and $n \in N_l(J) \leftrightarrow [[n, J], J] = 0$. If $A = \Phi E_{11} + \Phi E_{12} + \Phi E_{22}$ consists of all upper triangular $2 \times 2$ matrices over $\Phi$, then $[J, J] = \Phi E_{12}$, so $n = E_{12}$ lies in $N_m(J)$, yet $n \notin N_l(J)$ since $[[n, E_{11}], E_{11}] = -[E_{12}, E_{11}] = E_{12} \neq 0$. Note that $n$ falls in the trivial ideal $B = \Phi E_{12}$.

Another example is the algebra $J(Q, c)$ determined by a quadratic form $Q$; here $n \in N_m(J)$ $\Leftrightarrow [[J, J], n] = 0$ and $n \in N_l(J)$ $\Leftrightarrow [[n, J], J] = 0$. If $A = E_1 \oplus E_2$ consists of all upper triangular $2 \times 2$ matrices over $E$, then $[J, J] = E_2$, so $n = E_2$ lies in $N_m(J)$, yet $n \notin N_l(J)$ since $[[n, E_{11}], E_{11}] = -[E_{12}, E_{11}] = E_{12} \neq 0$. Note that $n$ falls in the trivial ideal $B = \Phi E_{12}$.

Without reference to middle nuclei we can establish

**THEOREM.** If $J$ is a semiprime linear Jordan algebra then $C(I) \subset C(J)$ for any ideal $I \lhd J$. More generally, $C(J) \subset C(J)$ as soon as $J \lhd J$ and $J$ contains no trivial ideals of $J$ invariant under all derivations of $J$ which map $J$ into itself.

**PROOF.** The first assertion follows from the second, since by a result of Slin'ko [4] if $J$ is semiprime so is any $I \lhd J$. Assume $J$ is $D$-semiprime in $\tilde{J}$ in the above sense; for convenience we may assume that $\tilde{J}$ is unital. We must show that if $c \in C(J)$ then $c \in C(\tilde{J}) = N_l(\tilde{J})$, i.e. $[c, \tilde{J}, \tilde{J}] = 0$.

All the derived ideals $J^{(n)}$ (where $J^{(0)} = J, J^{(n+1)} = U_{J(n)} J^{(n)}$) are invariant under the indicated derivations and remain ideals in $\tilde{J}$, as are their annihilators $J^{(n)}\perp$ (if $B \lhd J$ is invariant so is $B\perp = \{z \in J(z, B, \tilde{J}) = 0\}$, and $B\perp \lhd \tilde{J}$ since $\{z, B, \tilde{J}\} = \tilde{J}\{z, B, \tilde{J}\} - \{z, B, \tilde{J}\} \subset \tilde{J}\{z, B, \tilde{J}\} = 0$).

Moreover, $B = J \cap J^{(n)}\perp$ are solvable: $B^{(n)} \subset J^{(n)} \cap J^{(n)}\perp, B^{(n+1)} = 0$. If $J$ contains no trivial invariant ideals then it contains no solvable invariant ideals of $\tilde{J}$, so $B = 0$:

(15) if $J$ is $D$-semiprime in $\tilde{J}$ then $J \cap J^{(n)}\perp = 0$.

If $D(J^{(n)}) = 0$ then $D(J) \subset J^{(n+1)}\perp$ since $\{D(\tilde{J}), J^{(n+1)}, \tilde{J}\} \subset \{D(\tilde{J}), J^{(n)}, J^{(n)}\}$ (because $\{d, U_{xy}, \tilde{\alpha}\} = \{d, x, \{y, x, \tilde{\alpha}\}\} = \{d, U_{xy}, y\}$ where $x, y \in J^{(n)} \lhd \tilde{J} = D(\{J, J^{(n)}\} = \{J, D(J^{(n)}), J^{(n)}\} - \{J, J^{(n)}, D(J^{(n)})\} = 0$ (because $J^{(n)} \lhd \tilde{J}, D(J^{(n)}) = 0$), so from (15) we see

(16) if $J$ is $D$-semiprime in $\tilde{J}$ then $D(J^{(n)}) = 0 \Rightarrow D(\tilde{J}) = 0$

for any derivation of $\tilde{J}$ into $J$.

In particular, for $D = D_{c, J}$ as in (9) we see $[c, J, J] = 0$ (by $c \in N_l(J)$) implies $[c, \tilde{J}, \tilde{J}] = 0$, hence for $D = D_{c, J}$ we see $[c, J^2, \tilde{J}] = -[J^2, J, c] - [\tilde{J}, c, J^2]$ (by (4)) $\subset [c, \tilde{J}, J^2] + [J, c, J\tilde{J} + J\tilde{J}]$ (by (3) and linearized (6)) $\subset [c, \tilde{J}, J] + [J, c, J]$ (by $J \lhd \tilde{J}$) $= 0$ (by the above and $c \in N_m(J)$) implies $[c, \tilde{J}, \tilde{J}] = 0$ as desired.
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