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MIDDLE NUCLEUS = CENTER IN SEMIPRIME JORDAN ALGEBRAS

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ABSTRACT. A. A. Albert showed that the middle nucleus and center coincide for a simple Jordan algebra finite-dimensional over a field of characteristic $\neq 2$. E. Kleinfeld extended this to arbitrary simple Jordan algebras of characteristic $\neq 2$. Recently this result has played a crucial role in the structure theory of E. Zelmanov. In this note we extend the result to linear Jordan algebras with no derivation-invariant trivial ideals.

The *left, middle,* and *right nucleus* $N_l(A)$, $N_m(A)$, $N_r(A)$ of any nonassociative algebra A consists of the elements $n \in A$ with $[n, A, A] = 0$, $[A, n, A] = 0$, $[A, A, n] = 0$ respectively, where the *associator* is given by $[x, y, z] = (xy)z - x(yz)$. The *nucleus* $N(A)$ consists of the elements in all three nuclei simultaneously, and the *center* $C(A)$ consists of the nuclear elements which commute with A, $[n, A] = 0$. A *derivation* of A is a linear transformation D satisfying

$$D(xy) = D(x)y + xD(y) \tag{1}$$

and hence necessarily

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)] \tag{2}$$

as well. From this we see each nucleus is derivation-invariant, $D(N_i(A)) \subset N_i(A)$ for all D (the same is true of the center). When A is commutative we have

(3) $[x, y, z] = -[z, y, x]$,

(4) $[x, y, z] + [y, z, x] + [z, x, y] = 0$,

so $C(A) = N(A) = N_l(A) = N_r(A) \subset N_m(A)$. Our goal is to show conversely that for semiprime Jordan algebras the reverse inclusion holds as well. This has become important in Zelmanov's work [4], where expressions $[x_3, [x_1, x_2]^2, x_4]$ play a crucial role and one wants to know that if these vanish for all x_3, x_4 (i.e. $[x_1, x_2]^2$ is middle-nuclear) then $[x_1, x_2]^2$ is actually central.

We restrain our quadratic sympathies and work entirely with (nonunital) *linear Jordan algebras* J over a ring of scalars Φ containing $\frac{1}{2}$. Thus J has product xy satisfying the Jordan axioms

(5) $xy = yx$, i.e. $[x, y] = 0$,

(6) $(x^2y)x = x^2(yx)$, i.e. $[x^2, y, x] = 0$.

In addition to the left multiplication operator $L_x(z) = xz$, the U-operator

(7) $U_x(z) = (2L_x^2 - L_{x^2})(z)$

plays an important role even in the linear theory. It satisfies the identity

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(8) $L_yU_x + U_xL_y = U_{yx,x}$ $(U_{x,y} = U_{x+y} - U_x - U_y)$.
The *inner derivations* $D_{x,y}$ are defined by
(9) $D_{x,y}(z) = [L_x, L_y](z) = -[x, z, y]$;
any derivation D satisfies
(10) $D(U_xy) = U_{D(x),x}y + U_xD(y)$, $[D, D_{x,y}] = D_{D(x),y} + D_{x,D(y)}$.
A Jordan algebra is *semiprime* if it contains no *trivial* ideals B $(U_BB = 0)$ and is *D-semiprime* if it contains no nonzero trivial ideals B invariant under all derivations.

THEOREM. *The middle nucleus and center coincide, $N_m(J) = C(J)$, for any D-semiprime linear Jordan algebra J.*

PROOF. We must show $n \in N_m(J) \Rightarrow n \in N_l(J)$, i.e. $[J, n, J] = 0 \Rightarrow [n, J, J] = 0$. Following Albert [1] we introduce an ideal

$$B = [n, J, J] \tag{11}$$

which measures the failure of n to lie in $N_l(J)$. Here B is spanned by all associators $[n, x, y]$ for $x, y \in J$, where

$$[n, x, y] = [n, y, x] \tag{12}$$

since $0 = [n, x, y] + [x, y, n] + [y, n, x] = [n, x, y] - [n, y, x] + 0$ by (4), (3), and the definition of $n \in N_m(J)$. Thus $[n, x, y] = \frac{1}{2}([n, x, y] + [n, y, x])$ results from linearizing $[n, x, x]$, so

$$B \text{ is spanned by all } [n, x, x] \text{ for } x \in J. \tag{13}$$

To see B is actually an ideal, note JB is spanned by all $y[n, x, x] = -x[n, y, x] + [n, xy, x]$ (by (1), (9)) $= -x[n, x, y] + [n, xy, x]$ (by (12)) $= -\frac{1}{2}[n, x^2, y] + [n, xy, x]$ (by (1)) $\in [n, J, J] = B$.

Now by (13), $[n, B, J]$ is spanned by elements $[n, [n, x, x], y] = -[[n, x, x], y, n] - [y, n, [n, x, x]]$ (by (4)) $= D_{n,D_{n,x}(x)}(y) + 0$ (by (9)) $= \{[D_{n,x}, D_{n,x}] - D_{D_{n,x}(n),x}\}(y)$ (by (10)) $= 0$ (since $D_{J,J}(n) = 0$), so

$$[n, B, J] = 0. \tag{14}$$

Then $0 = [n, B, J] \supset [n, U_BJ, J] = U_{[n,B,J],B}J + U_B[n, J, J]$ (by (9), (10)) $= 0 + U_BB$ implies $U_BB = 0$. If J is semiprime then $B = [n, J, J] = 0$ by (11) and $n \in N_l(J)$.

If J is merely D-semiprime we must work a little harder. We claim $C = [N_m(J), J, J] = \sum[n, J, J]$ is a trivial D-invariant ideal. C is an ideal since each $[n, J, J]$ is by (11), and it is D-invariant by (2) and the fact that $N_m(J)$ is D-invariant. As in (14) we have

$$[n, C, J] = 0 \tag{14'}$$

for any $n \in N_m(J)$ since by (12) this is spanned by all $[n, [n', x, x], y] = [[n, n', y], x, x] + [n', [n, x, y], x] + [n', x, [n, x, y]]$ (by (9), (2)) $= 0 + 2[n', [n, x, y], x]$ (by (12) and $n' \in N_m(J)$) $= 2\{n'[n, y, x]\}x - 2n'\{[n, x, y]x\}$ (by (12)) $= [n, x^2, n'y] - [n, n'y, x^2]$ (by (1), (9)) $= 0$ (by (12)). Then $0 = [n, C, J] \supset [n, CJ, J] = [n, C, J]J + C[n, J, J]$ (by (1)) $= 0 + C[n, J, J]$ (by (14')); this holds for all $n \in N_m(J)$, so $0 = CC$ and C is trivial as well as D-invariant. If J is D-semiprime this forces $C = 0$, so $[N_m(J), J, J] = 0$ and $N_m(J) \subset N_l(J)$. □

It is easy to give examples to show that the semiprimeness hypothesis is needed. If we take $J = F/K$ for F the free Jordan algebra on x, y, K the ideal generated by all $[a, y, b]$ for $a, b \in F$, then $n = y + K$ is middle-nuclear in J by construction but is not central: $[n, J, J] \neq 0$ since $[y, x, x] \notin K$ (K is graded, vanishing in total degrees < 3, and in degree x^2y^1 it is spanned by $[x, y, x] = 0$, so $[y, x, x] \notin K$).

We can also easily give special examples $J = A^+$ for A an associative algebra. Here $n \in N_m(J) \Leftrightarrow [[J, J], n] = 0$ and $n \in N_l(J) \Leftrightarrow [[n, J], J] = 0$. If $A = \Phi E_{11} + \Phi E_{12} + \Phi E_{22}$ consists of all upper triangular 2×2 matrices over Φ, then $[J, J] = \Phi E_{12}$, so $n = E_{12}$ lies in $N_m(J)$, yet $n \notin N_l(J)$ since $[[n, E_{11}], E_{11}] = -[E_{12}, E_{11}] = E_{12} \neq 0$. Note that n falls in the trivial ideal $B = \Phi E_{12}$.

Another example is the algebra $J(Q, c)$ determined by a quadratic form Q; here $J = \Phi c \bigoplus V$ where $[v_1, v_2, v_3] = Q(v_1, v_2)v_3 - Q(v_2, v_3)v_1$, so $B = \operatorname{Rad} Q \subset N_m$ but $B \not\subset N_r$ if $Q \neq 0$.

Without reference to middle nuclei we can establish

THEOREM. *If J is a semiprime linear Jordan algebra then $C(I) \subset C(J)$ for any ideal $I \lhd J$. More generally, $C(J) \subset C(\tilde{J})$ as soon as $J \lhd \tilde{J}$ and J contains no trivial ideals of $\tilde{J}$ invariant under all derivations of $\tilde{J}$ which map J into itself.*

PROOF. The first assertion follows from the second, since by a result of Slin'ko [4] if J is semiprime so is any $I \lhd J$. Assume J is D-semiprime in $\tilde{J}$ in the above sense; for convenience we may assume that $\tilde{J}$ is unital. We must show that if $c \in C(J)$ then $c \in C(\tilde{J}) = N_l(\tilde{J})$, i.e. $[c, \tilde{J}, \tilde{J}] = 0$.

All the derived ideals $J^{(n)}$ (where $J^{(0)} = J$, $J^{(n+1)} = U_{J^{(n)}}J^{(n)}$) are invariant under the indicated derivations and remain ideals in $\tilde{J}$, as are their annihilators $(J^{(n)})^\perp$ (if $B \lhd J$ is invariant so is $B^\perp = \{z \in \tilde{J} | \{z, B, \tilde{J}\} = 0\}$, and $B^\perp \lhd \tilde{J}$ since $\{\tilde{J}z, B, \tilde{J}\} = \tilde{J}\{z, B, \tilde{J}\} - \{z, B, \tilde{J}\tilde{J}\} + \{z, \tilde{J}B, \tilde{J}\} \subset \tilde{J}\{z, B, \tilde{J}\} = 0$). Moreover, $B = J \cap J^{(n)\perp}$ are solvable: $B^{(n)} \subset J^{(n)} \cap J^{(n)\perp}$, $B^{(n+1)} = 0$. If J contains no trivial invariant ideals then it contains no solvable invariant ideals of $\tilde{J}$, so $B = 0$:

$$\text{if } J \text{ is } D\text{-semiprime in } \tilde{J} \text{ then } J \cap J^{(n)\perp} = 0. \tag{15}$$

If $D(J^{(n)}) = 0$ then $D(\tilde{J}) \subset J^{(n+1)\perp}$ since $\{D(\tilde{J}), J^{(n+1)}, \tilde{J}\} \subset \{D(\tilde{J}), J^{(n)}, J^{(n)}\}$ (because $\{d, U_x y, \tilde{a}\} = \{d, x, \{y, x, \tilde{a}\}\} - \{d, U_x\tilde{a}, y\}$ where $x, y \in J^{(n)} \lhd \tilde{J}$) $= D(\{\tilde{J}, J^{(n)}, J^{(n)}\}) - \{\tilde{J}, D(J^{(n)}), J^{(n)}\} - \{\tilde{J}, J^{(n)}, D(J^{(n)})\} = 0$ (because $J^{(n)} \lhd \tilde{J}, D(J^{(n)}) = 0$), so from (15) we see

$$\begin{aligned}&\text{if } J \text{ is } D\text{-semiprime in } \tilde{J} \text{ then } D(J^{(n)}) = 0 \Rightarrow D(\tilde{J}) = 0\\ &\text{for any derivation of } \tilde{J} \text{ into } J.\end{aligned} \tag{16}$$

In particular, for $D = D_{c,J}$ as in (9) we see $[c, J, J] = 0$ (by $c \in N_l(J)$) implies $[c, \tilde{J}, J] = 0$, hence for $D = D_{c,\tilde{J}}$ we see $[c, J^2, \tilde{J}] = -[J^2, \tilde{J}, c] - [\tilde{J}, c, J^2]$ (by (4)) $\subset [c, \tilde{J}, J^2] + [J, c, J\tilde{J} + \tilde{J}J]$ (by (3) and linearized (6)) $\subset [c, \tilde{J}, J] + [J, c, J]$ (by $J \lhd \tilde{J}$) $= 0$ (by the above and $c \in N_m(J)$) implies $[c, \tilde{J}, \tilde{J}] = 0$ as desired. $\square$

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