AN INFINITE CLASS OF PERIODIC SOLUTIONS OF PERIODICALLY PERTURBED DUFFING EQUATIONS AT RESONANCE

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ABSTRACT. In this paper, by using a generalised form of the Poincaré-Birkhoff Theorem, we demonstrate that the Duffing equation

\[ \frac{d^2 x}{dt^2} + g(x) = p(t) \quad (\equiv p(t + 2\pi)) \]

may also admit an infinite number of \(2\pi\)-periodic solutions even in a resonance case.

1. In this paper we study the existence problem of periodic solutions for the Duffing equation

\[ \frac{d^2 x}{dt^2} + g(x) = p(t) \]

where \(g(x), p(t) \in C(R, R)\) and \(p(t)\) is \(2\pi\)-periodic.

In the super-linear case, that is,

\[ \lim_{|x| \to \infty} [x^{-1}g(x)] = \infty, \]

it has been proved in a recent paper by W. Y. Ding that (1.1) has infinitely many \(2\pi\)-periodic solutions [1].

D. E. Leach proved the existence and uniqueness of \(2\pi\)-periodic solution of (1.1) under the following Loud condition

\[ m^2 < \lambda \leq g'(x) \leq \mu < (m + 1)^2 \quad (g(0) = 0), \]

with a given integer \(m \geq 0\) and two such constants \(\lambda\) and \(\mu\) [2]. R. Reissig proved the existence of periodic solutions of (1.1) under a weaker condition [3]

\[ m^2 < \lambda \leq x^{-1}g(x) \leq \mu < (m + 1)^2, \quad |x| \geq a > 0. \]

The crucial point of the conditions above is to exclude the resonance cases. At resonance, (1.1) may have no bounded solution [6]; therefore, the existence problem of periodic solutions challenges attention. A. C. Lazer and D. E. Leach, and L. Césari succeeded to prove the existence of \(2\pi\)-periodic solutions of (1.1) at a point of resonance provided that \(h(x) = g(x) - m^2x\) is bounded and some other additional conditions are assumed [4, 5]. In a recent paper, the present author resolved the existence problem of \(2\pi\)-periodic solutions of (1.1) under a weakened version of (1.2) [6]:

\[ m^2 \leq g'(x) \leq (m + 1)^2 \quad (g(0) = 0). \]

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In this paper, by using a generalized form of the Poincaré-Birkhoff Theorem due to W. Y. Ding, we will demonstrate that (1.1) may also admit an infinite number of $2\pi$-periodic solutions even in a resonance case. This shows once more the complexity of (1.1) at resonance.

2. We need the following hypotheses:

(H1) Let $g(x) \in C'(R, R)$, and let $K$ be a positive constant, such that

$$|g'(x)| \leq K, \quad x \in R;$$

(H2) There exist two constants $A_0 > 0$ and $M_0 > 0$, such that

$$x^{-1}g(x) \geq A_0, \quad |x| \geq M_0.$$

Then we consider the auxiliary equation

$$\frac{d^2z}{dt^2} + g(z) = 0$$

and its equivalent system

$$\begin{align*}
\frac{dz}{dt} &= w, \\
\frac{dw}{dt} &= -g(z).
\end{align*}$$

This is a planar autonomous system whose orbits are curves determined by the following equation

$$V(z, w) \equiv \frac{1}{2}w^2 + G(z) = c,$$

where $G(z) = \int_0^z g(u) \, du$ and $c$ is a parameter.

The hypothesis (H2) obviously implies

$$\lim_{|z| \to \infty} G(z) = \infty \quad \text{and} \quad \lim_{|z| + |w| \to \infty} V(z, w) = \infty.$$

It follows that the set $V^{-1}(c)$ is compact. Furthermore, (H2) yields that there exist constants $C_0 > 0$ and $A_1 > 0$ such that if $c \geq c_0$, then

$$w^2 + zg(z) \geq A_1(z^2 + w^2), \quad (z, w) \in V^{-1}(c).$$

Note that the left-hand member of the last inequality is just the directional derivative of $V(z, w)$ along the vector $(z, w)$. Therefore, there is no critical point in $V^{-1}(c)$ for $c \geq c_0$, and $V^{-1}(c)$ is a compact one-dimensional manifold. Furthermore $V^{-1}(c)$ is star-shaped about the origin. We have thus proved

**Lemma 2.1.** If (H2) holds, then $V^{-1}(c)$ is a closed curve for $c \geq c_0$ which is star-shaped about the origin.

In the sequel, we will denote the curve $V^{-1}(c)$ by $\Gamma_c$. It follows from Lemma 2.1 that each curve $\Gamma_c$ ($c \geq c_0$) intersects the z-axis at two points: $(h(c), 0)$ and $(-h_1(c), 0)$, where $h(c) > 0$ and $h_1(c) > 0$ are uniquely determined by the formula

$$G(h(c)) = G(-h_1(c)) = c.$$

Let $(z(t), w(t))$ be any solution of (2.1') whose orbit is $\Gamma_c$ ($c \geq c_0$). Clearly, this solution is periodic. Let $\tau(c)$ denote the least positive period of this solution. It follows from the equation (2.2) that

$$\tau(c) = \sqrt{2} \int_{-h_1(c)}^{h(c)} \frac{du}{\sqrt{c - G(u)}}.$$
To obtain the desired result, we need another hypothesis:

\((H_3)\) There exist a constant \(\alpha > 0\), an integer \(m > 0\), and two sequences \(\{a_k\}\) and \(\{b_k\}\), such that \(a_k \to \infty\) and \(b_k \to \infty\) as \(k \to \infty\); and moreover

\[
\tau(a_k) < \frac{2\pi}{m} - \alpha, \quad \tau(b_k) > \frac{2\pi}{m} + \alpha.
\]

Now we are in a position to state the main result of this paper.

**Theorem 2.1.** Assume \((H_1)-(H_3)\) hold. Then equation (1.1) has infinitely many \(2\pi\)-periodic solutions.

A concrete example for applications of this theorem will be given in §5.

3. To prove Theorem 2.1, we briefly restate a generalized form of the Poincaré-Birkhoff fixed point theorem in Theorem A below [7].

Let \(D\) denote an annular region in the \((x, y)\)-plane. The boundary of \(D\) consists of two simple closed curves: the inner boundary curve \(C_1\) and the outer boundary curve \(C_2\). Let \(D_1\) denote the simple connected open set bounded by \(C_1\). Consider an area-preserving mapping \(T: \mathbb{R}^2 \to \mathbb{R}^2\). Suppose that \(T(D) \subset \mathbb{R}^2 - \{0\}\), where 0 is the origin. Let \((\gamma, \theta)\) be the polar coordinate of \((x, y)\), that is, \(x = \gamma \cos \theta, y = \gamma \sin \theta\). Assume the restriction \(T|_D\) is given by

\[
\gamma^* = f(\gamma, \theta), \quad \theta^* = \theta + g(\gamma, \theta),
\]

where \(f\) and \(g\) are continuous in \((\gamma, \theta)\), and \(2\pi\)-periodic in \(\theta\).

**Theorem A.** Besides the above-mentioned assumptions, we assume that

(i) \(C_1\) is star-shaped about the origin;
(ii) \(0 \in T(D_1)\);
(iii) \(g(\gamma, \theta) > 0\) \((< 0)\), \((\gamma \cos \theta, \gamma \sin \theta) \in C_1\);
\[
g(\gamma, \theta) < 0\) \((> 0)\), \((\gamma \cos \theta, \gamma \sin \theta) \in C_2\).

Then \(T\) has at least two fixed points in \(D\).

Now, let \(\Gamma_{a_k}\) and \(\Gamma_{b_k}\) be the curves given by Lemma 2.1, where the specified parameters \(a_k, b_k \geq c_0\) are given by \((H_3)\), for \(k \geq n_0\). We may rearrange \(\{a_k\}\) and \(\{b_k\}\), if necessary, so that \(a_k < b_k < a_{k+1}\) for \(k \geq n_0\). Then, \(\Gamma_{a_k}\) and \(\Gamma_{b_k}\) bound an annular region \(A_k\), and \(\Gamma_{b_k}\) and \(\Gamma_{a_{k+1}}\) bound another annular region \(B_k\), for \(k \geq n_0\).

Let \(T: \mathbb{R}^2 \to \mathbb{R}^2\) be the Poincaré mapping induced by equation (1.1). It is well known that each fixed point of \(T\) corresponds to a \(2\pi\)-periodic solution of (1.1). In the following section, we will apply Theorem A to show that \(T\) has at least two fixed points in each \(A_k\) and \(B_k\) for sufficiently large \(k\). As a consequence, (1.1) has an infinite class of \(2\pi\)-periodic solutions.

4. Consider the equivalent system of (1.1),

\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -g(x) + p(t).
\]

Let \((\bar{x}(t, x, y), \bar{y}(t, x, y))\) be the solution of (4.1) through the initial point \((\bar{x}(0), \bar{y}(0)) = (x, y)\). It is not hard to show that every such solution exists on the whole \(t\)-axis under condition \((H_1)\). Then the Poincaré map \(T: \mathbb{R}^2 \to \mathbb{R}^2\) is well defined by

\[(x, y) \mapsto (\bar{x}(2\pi, x, y), \bar{y}(2\pi, x, y)).\]
It is well known that $T$ is an area-preserving homeomorphism.

By applying the transform $x(t) = \gamma(t) \cos \theta(t)$, $y(t) = \gamma(t) \sin \theta(t)$ to (4.1), we get the equations for $\gamma(t)$ and $\theta(t)$,

\begin{align*}
\frac{d\gamma}{dt} &= \gamma \cos \theta \cdot \sin \theta - g(\gamma \cos \theta) \sin \theta + p(t) \sin \theta, \\
\frac{d\theta}{dt} &= -\sin^2 \theta - \frac{1}{\gamma} (g(\gamma \cos \theta) \cos \theta - p(t) \cos \theta),
\end{align*}

whenever $\gamma \neq 0$.

Let $(\gamma(t, \gamma, \theta), \theta(t, \gamma, \theta))$ be the solution of (4.2) through the initial point $(\gamma(0), \theta(0)) = (\gamma, \theta)$. Then the map $T$ can also be written in the polar coordinate form

\begin{align*}
\gamma^* &= \gamma(2\pi, \gamma, \theta), & \theta^* &= \theta(2\pi, \gamma, \theta) + 2l \pi,
\end{align*}

where $l$ is an arbitrary integer. It can be easily seen that if $(\gamma, \theta)$ is such that

\begin{align*}
\gamma(t, \gamma, \theta) > 0, & & t \in [0, 2\pi],
\end{align*}

then $\theta(2\pi, \gamma, \theta)$ is well defined and continuous in $(\gamma, \theta)$, and moreover,

\begin{align*}
\theta(2\pi, \gamma, \theta + 2\pi) &= \theta(2\pi, \gamma, \theta) + 2\pi.
\end{align*}

Next, we take the transform $z(t) = \rho(t) \cos \phi(t)$, $w(t) = \rho(t) \sin \phi(t)$ for system (2.1'). Then the resulting equations for $\rho(t)$ and $\phi(t)$ are

\begin{align*}
\frac{d\rho}{dt} &= \rho \cos \phi \cdot \sin \phi - g(\rho \cos \phi) \sin \phi, \\
\frac{d\phi}{dt} &= -\sin^2 \phi - \frac{1}{\rho} g(\rho \cos \phi) \cos \phi.
\end{align*}

Let $(\rho(t, \rho, \phi), \phi(t, \rho, \phi))$ be the solution of (4.6) through the initial point $(\rho(0), \phi(0)) = (\rho, \phi)$.

**Lemma 4.1.** Let $\Phi(\rho, \phi) = \phi(2\pi, \rho, \phi) - \phi$, and let $\beta = \min\{2\pi, m\alpha A_1\}$, where $m$, $\alpha$ and $A_1$ are given in (H$_3$) and (2.3). Then we have

\begin{align*}
\Phi(\rho, \phi) &\leq -2m\pi - \beta, & \rho \cos \phi, \rho \sin \phi \in \Gamma_{ak};
\Phi(\rho, \phi) &\geq -2m\pi + \beta, & \rho \cos \phi, \rho \sin \phi \in \Gamma_{bk}.
\end{align*}

**Proof.** Let $(z, w) = (\rho \cos \phi, \rho \sin \phi) \in \Gamma_{ak}$. Consider the solution $(\rho(t, \rho, \phi), \phi(t, \rho, \phi))$ of (4.6). It follows from (2.3) and the second equation of (4.6) that

\begin{align*}
\phi'(t, \rho, \phi) &\leq -A_1,
\end{align*}

provided that $\rho$ is sufficiently large.

Since the solution $(\tilde{z}(t, z, w), \tilde{w}(t, z, w))$ of (2.1') has the least period $\tau(a_k)$, we see that the time in which $\tilde{\phi}(t)$ has a decrement $2\pi$ is just $\tau(a_k)$. Write

\begin{align*}
\Phi(\rho, \phi) &= \phi(2\pi, \rho, \phi) - \phi = \phi(2\pi) - \phi(0) = -2l \pi - \sigma,
\end{align*}

where $l \geq 0$ is an integer, and $0 \leq \sigma < 2\pi$. Let $t_{\sigma}$ denote the time in which $\tilde{\phi}(t)$ decreases from $\phi - 2l \pi$ to $\phi - 2l \pi - \sigma$. Then we have

\begin{align*}
l \cdot \tau(a_k) + t_{\sigma} &= 2\pi.
\end{align*}
Since $0 \leq t_\sigma < \tau(a_k)$, we obtain
\[ 2\pi = l \cdot \tau(a_k) + t_\sigma < (l + 1)\tau(a_k) \leq (l + 1)\left(\frac{2\pi}{m} - \alpha\right). \]

It follows that $l \geq m$. If $l \geq m + 1$, we have
\[ (4.9) \quad \Phi(\rho, \varphi) \leq -2l\pi \leq -(m + 1)\pi. \]

Now, assume $l = m$. Then we have
\[ (4.10) \quad t_\sigma = 2\pi - m \cdot \tau(a_k) \geq 2\pi - m\left(\frac{2\pi}{m} - \alpha\right) = m\alpha. \]

By (4.8) and (4.10), we obtain
\[ -\sigma = \int_{l \cdot \tau(a_k) + t_\sigma} \varphi'(t, \rho, \varphi) \, dt \leq -A_1 t_\sigma \leq -m\alpha A_1. \]

Thus we have
\[ (4.11) \quad \Phi(\rho, \varphi) = -2l\pi - \sigma \leq -2m\pi - m\alpha A_1. \]

Combining (4.9) and (4.11) yields the first inequality of (4.7).

The second inequality of (4.7) can be proved in a similar way. The proof of Lemma 4.1 is then completed.

**Lemma 4.2.** Let $\gamma(\gamma, \theta) = \bar{d}(2\pi, \gamma, \theta) - \theta$. Then there exists $\gamma_0 > 0$ such that, for $\gamma \geq \gamma_0$,
\[ |\Theta(\gamma, \theta) - \Phi(\gamma, \theta)| = |\bar{d}(2\pi, \gamma, \theta) - \varphi(2\pi, \gamma, \theta)| \leq \beta. \]

**Proof.** Let $(\bar{x}(t, x, y), \bar{w}(t, x, y))$ be the solution of (2.1') through the initial point $(\bar{x}(0), \bar{w}(0)) = (x, y)$. Let
\[ \begin{align*}
  u(t) &= u(t, x, y) = \bar{x}(t, x, y) - \bar{x}(t, x, y), \\
  v(t) &= v(t, x, y) = \bar{y}(t, x, y) - \bar{w}(t, x, y).
\end{align*} \]

Then we have
\[ \frac{du}{dt} = v, \quad \frac{dv}{dt} = p(t) - g'(\sigma(t))u, \]
where $\sigma(t) = \bar{x}(t) + \lambda(t)(\bar{x}(t) - \bar{x}(t)), 0 \leq \lambda(t) \leq 1$.

Let $\eta(t) = (u^2(t) + v^2(t))^{1/2}$. Then we have
\[ \eta \frac{d\eta}{dt} = uv + p(t)v - g'(\sigma(t))uv. \]

It follows from $(H_1)$ that
\[ (4.13) \quad \left| \frac{d\eta}{dt} \right| \leq \frac{1}{2}(1 + K)\eta + B, \]

where $B$ is a bound of $|p(t)|$ in $[0, 2\pi]$. The differential inequality (4.13) together with $\eta(0) = 0$ yields
\[ (4.14) \quad \eta(t) \leq \frac{2B}{K + 1}(e^{(K+1)\pi} - 1) \equiv H_0, \]
for $t \in [0, 2\pi]$. 

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Next write \( \psi(t) = \psi(t, \gamma, \theta) = \varphi(t, \gamma, \theta) - \varphi(t, \gamma, \theta) \), where \((\gamma, \theta)\) is the polar coordinate of \((x, y)\), i.e., \((\gamma \cos \theta, \gamma \sin \theta) = (x, y)\). It is clear that if \(|\psi(t)| < \pi\), then \(\psi(t)\) is just the angle between the vectors \((x(t), y(t))\) and \((z(t), w(t))\). By the law of cosines, we have

\[
\cos \psi(t) = \frac{\gamma^2(t) + \rho^2(t) - \eta^2(t)}{2\gamma(t)\rho(t)} \geq 1 - \frac{H_0^2}{2\gamma(t)\rho(t)}.
\]

On the other hand, we have \(\gamma(t) \geq \rho(t) - \eta(t) \geq \rho(t) - H_0\). Therefore, under the assumption that \(|\psi(t)| < \pi\) and \(\rho(t) - H_0 > 0\), we have

\[
\cos \psi(t) \geq 1 - \frac{H_0^2}{2\rho(t)(\rho(t) - H_0)}.
\]

Note that

\[
F(\gamma, \varphi) = \inf_{0 \leq t \leq 2\pi} \tilde{\varphi}(t) = \inf_{0 \leq t \leq 2\pi} \tilde{\varphi}(t, \gamma, \varphi)
\]

becomes arbitrarily large if \(\gamma\) is sufficiently large. It follows that there is a constant \(\gamma_0 > 0\) such that, for \(\gamma \geq \gamma_0\) and \(t \in [0, 2\pi]\),

\[
\rho(t) - H_0 > 0; \quad \frac{H_0^2}{2\rho(t)(\rho(t) - H_0)} < 1 - \cos \delta,
\]

where \(\delta = \min(\pi/2, \beta)\). From (4.15) and (4.16), we conclude that if \(|\psi(t)| < \pi\) and \(t \in [0, 2\pi]\), then the inequality

\[
|\psi(t)| < \delta
\]

holds.

Since \(\psi(0) = 0\) and \(\psi(t)\) varies continuously as \(t\) increases from 0 to \(2\pi\), we can see that (4.17) holds for any \(t \in [0, 2\pi]\). In particular, we have

\[
|\psi(2\pi)| = |\phi(2\pi, \gamma, \theta) - \varphi(2\pi, \gamma, \theta)| < \delta \leq \beta,
\]

for \(\gamma \geq \gamma_0\). This proves Lemma 4.2.

**PROOF OF THEOREM 2.1.** Let \(c_1 > c_0\) be so large that \((\gamma, \theta) \in \Gamma_{c_1}\) with \(c \geq c_1\) implies \(\gamma \geq \gamma_0\), where \(\gamma_0\) is specified in Lemma 4.2. There is no loss of generality to assume \(a_k \geq c_1\) for \(k \geq n_0\). It follows that \(\gamma(t, \gamma, \theta) \geq \rho(t) - H_0 > 0\), \(t \in [0, 2\pi]\), provided that \((\gamma, \theta) \in \Lambda_k\) for \(k \geq n_0\).

Thus the restriction \(T|A_k\) can be written in (4.3), where the function \(\bar{\phi}(2\pi, \gamma, \theta)\) is continuous on \(A_k\) and satisfies the relation (4.4). Now, we put the integer \(l = m\) in (4.3). Then (4.3) can be rewritten in the form of

\[
\gamma^* = \tilde{\gamma}(2\pi, \gamma, \theta), \quad \theta^* = \theta + \Theta_1(\gamma, \theta),
\]

with \(\Theta_1(\gamma, \theta) = \Theta(\gamma, \theta) - 2m\pi\). By Lemma 4.2, we obtain

\[
|\Theta_1(\gamma, \theta) - \Phi(\gamma, \theta) - 2m\pi| < \beta,
\]

which together with (4.7) yields

\[
\Theta_1(\gamma, \theta) < 0, \quad (\gamma \cos \theta, \gamma \sin \theta) \in \Gamma_{a_k};
\]

\[
\Theta_1(\gamma, \theta) > 0, \quad (\gamma \cos \theta, \gamma \sin \theta) \in \Gamma_{b_k},
\]

for \(k \geq n_0\), where \(n_0\) is large enough.

This proves the validity of condition (iii) of Theorem A for the restriction \(T|A_k\).
(k ≥ n₀). Since γ ≥ γ₀ implies 2γπ ≥ 2π — H₀ > 0, so the condition (ii) of Theorem A can be easily verified. By Lemma 2.1, the condition (i) of Theorem A also holds. Therefore, we can apply Theorem A to ensure the existence of at least two fixed points of T in A_k (k ≥ n₀). This means that (1.1) has at least two 2π-periodic solutions with initial points in A_k. In the same way, we can prove that T has at least two fixed points in B_k which correspond to two 2π-periodic solutions of (1.1). Since each periodic solution of (1.1) is bounded by a Γₜₙ, so the above-specified 2π-periodic solutions of (1.1) constitute an infinite class.

The proof of Theorem 2.1 is thus completed.

5. Finally, we give a concrete example for applications of Theorem 2.1.

Let m be a positive integer, and let

\[ g(x) = x \left[ m^2 + \frac{1}{10} \cos \log(1 + x^2) - \frac{x^2}{10(1 + x^2)} \right]. \]

A direct calculation yields

\[ m^2 - \frac{1}{2} \leq g'(x) \leq m^2 + \frac{1}{2}. \]

It follows that (H₁) and (H₂) hold for g(x).

Consider the auxiliary equation

\[ \frac{d^2 z}{dt^2} + g(z) = 0. \] (5.1)

According to (2.2), we have

\[ V(z, w) = \frac{1}{2} w^2 + \bar{G}(z) = c, \]

where

\[ \bar{G}(z) = \int_0^z g(u) \, du = \frac{1}{2} z^2 \left[ m^2 + \frac{1}{10} \cos \log(1 + z^2) \right]. \]

Take h = h(c) > 0 such that \( \bar{G}(h(c)) = c \). Since \( \bar{G}(z) \) is an even function, we have

\[ \tau(c) = 2\sqrt{2} \int_0^{h(c)} \frac{du}{\sqrt{c - \bar{G}(u)}} = \frac{4}{\sqrt{Q(h)}} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2 Q(h) Q(h)}}. \]

where

\[ Q(x) = m^2 + \frac{1}{10} \cos \log(1 + x^2). \]

Let \( \alpha_k = e^{2k\pi} - 1, \beta_k = e^{(2k+1)\pi} - 1, a_k = \bar{G}(\alpha_k) \) and \( b_k = \bar{G}(\beta_k) \). Then

\[ a_k < b_k \quad (k = 1, 2, \ldots) \]

and

\[ \lim_{k \to \infty} a_k = \lim_{k \to \infty} b_k = \infty. \]

Since the inequalities

\[ 0 < \frac{Q(\alpha_k \xi)}{Q(\alpha_k)} \leq 1, \quad \frac{Q(\beta_k \xi)}{Q(\beta_k)} \geq 1 \]
hold for $\xi \in [0, 1]$, we have
\[
\tau(a_k) = \frac{4}{\sqrt{Q(\alpha_k)}} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2 Q(\alpha_k \xi) / Q(\alpha_k)}}
\leq \frac{4\sqrt{10}}{\sqrt{10m^2 + 1}} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2}} \leq \frac{2\pi}{m} - \bar{\alpha},
\]
and
\[
\tau(b_k) = \frac{4}{\sqrt{Q(\beta_k)}} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2 Q(\beta_k \xi) / Q(\beta_k)}}
\geq \frac{4\sqrt{10}}{\sqrt{10m^2 - 1}} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi^2}} \geq \frac{2\pi}{m} + \bar{\alpha},
\]
where
\[
\bar{\alpha} = \frac{2\pi}{m} \cdot \min \left\{ 1 - \frac{\sqrt{10m}}{\sqrt{10m^2 + 1}}, \frac{\sqrt{10m}}{\sqrt{10m^2 - 1}} - 1 \right\}.
\]
This proves that (H3) is also valid for $\bar{g}(x)$.

Therefore, the function $\bar{g}(x)$ satisfies all the assumptions of Theorem 2.1, and the corresponding equation
\[
\frac{d^2 z}{dt^2} + \bar{g}(x) = p(t)
\]
has infinitely many $2\pi$-periodic solutions for any $2\pi$-periodic function $p(t) \in C(R, R)$.

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