ON DINI'S THEOREM AND A METRIC ON $C(X)$ TOPOLOGICALLY EQUIVALENT TO THE UNIFORM METRIC

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ABSTRACT. Let $X$ be a compact metric space and let $UC(X)$ denote the u.s.c. real valued functions on $X$. Let $\tau$ be a topology on $UC(X)$. $\Omega \subset UC(X)$ is called a Dini class of functions induced by $\tau$ if (1) $\Omega$ is $\tau$-closed, (2) $C(X) \subset \Omega$, (3) for each $h \in \Omega$ whenever $\{h_n\}$ is a decreasing sequence of u.s.c. functions convergent pointwise to $h$, then $\{h_n\}$ $\tau$-converges to $h$. By Dini's theorem the topology of uniform convergence on $UC(X)$ induces $C(X)$ as its Dini class of functions. As a main result, when $X$ is locally connected we show that the hyperspace topology on $UC(X)$ obtained by identifying each u.s.c. function with the closure of its graph induces a larger Dini class of functions than $C(X)$, even though the restriction of this topology to $C(X)$ agrees with the topology of uniform convergence.

Let $X$ be a compact metric space and let $C(X)$ and $UC(X)$ denote the continuous and upper semicontinuous real valued functions on $X$, respectively. If $f \in C(X)$ and $\{h_n\}$ is a decreasing sequence of u.s.c. functions on $X$ convergent pointwise to $f$, then Dini's theorem [10] says that the convergence must be uniform. Moreover, if $\{f_n\}$ is a sequence in $C(X)$ convergent uniformly to an u.s.c. function $h$, then $h$ must be in $C(X)$. Thus, the class $C(X)$, viewed as a subclass of $UC(X)$, is induced by the topology of uniform convergence on $UC(X)$ in the following sense.

DEFINITION. Let $X$ be a compact metric space and let $\Omega \subset UC(X)$. If $\tau$ is a topology on $UC(X)$, then $\Omega$ is called a Dini class of functions induced by $\tau$ if

1. $\Omega$ is $\tau$-closed,
2. $C(X) \subset \Omega$,
3. for each $h \in \Omega$ whenever $\{h_n\}$ is a decreasing sequence of u.s.c. functions convergent pointwise to $h$, then $\{h_n\}$ $\tau$-converges to $h$.

It follows from the definition that $\tau$ can induce at most one Dini class. To see this, suppose that $\Omega$ and $\Omega'$ are both Dini classes of $\tau$. Let $f \in \Omega$ be arbitrary. Since each u.s.c. function on $X$ is a pointwise limit of a decreasing sequence of continuous functions [6],

$$f \in \overline{cl}_{\tau} C(X) \subset \overline{cl}_{\tau} \Omega' = \Omega'.$$

Standard topologies on $UC(X)$ seem to induce nice Dini classes of functions. For example, let $\mu$ be a regular Borel measure on $X$. We first consider the convergence in measure topology on $UC(X)$. Recall that $\{h_n\}$ is said to converge in measure to

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$h$ if for each $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$,

$$\mu \{ x : |h_n(x) - h(x)| > \epsilon \} < \epsilon.$$  

It is well known [7] that convergence in this sense is described by a pseudometric on the $\mu$-measurable functions on $X$:

$$\delta_1(f, g) = \int_X \frac{|f - g|}{1 + |f - g|} \, d\mu.$$  

The $\delta_1$-topology induces $UC(X)$ as its Dini class. Conditions (1) and (2) hold trivially and condition (3) follows from Egoroff’s theorem. In the same context define the $L_1(\mu)$ pseudometric $\delta_2$: $UC(X) \times UC(X) \rightarrow [0, \infty]$ by

$$\delta_2(f, g) = \int_X |f - g| \, d\mu.$$  

We claim that $\delta_2$ induces $L_1(\mu) \cap UC(X)$, the u.s.c. functions that are $\mu$-integrable, as its Dini class. Condition (1) holds because $L_1(\mu)$ is closed under addition, condition (2) holds because $X$ is compact, and condition (3) follows from the Lebesgue dominated convergence theorem using as a majorant $g: X \rightarrow [0, \infty)$ defined by

$$g(z) = \max \left\{ \max_{x \in X} h_1(x), -\min \{h(z), 0\} \right\}.$$  

In this paper we shall focus our attention on two hyperspace topologies on $UC(X)$ considered in [2]. In short we identify members of $UC(X)$ with certain closed subsets of $X \times \mathbb{R}$ and apply the Hausdorff metric. We review the essential features of this construction. Let $Y$ be a metric space with metric $\rho$ and for each $y$ in $Y$ let $B_\lambda[y]$ denote the closed ball of radius $\lambda$ about $y$. If $K$ is a closed subset of $Y$, the $\lambda$-parallel body of $K$, denoted by $B_\lambda[K]$, is the set $\bigcup_{y \in K} B_\lambda[y]$. If $C$ and $K$ are closed sets in $Y$ we define the Hausdorff distance of $C$ from $K$ to be

$$D(C, K) = \inf \{ \lambda : B_\lambda[K] \supset C \text{ and } B_\lambda[C] \supset K \}.$$  

We refer the reader to Castaing and Valadier [3] or Nadler [9] for further information. Now let $X$ be a metric space with metric $d$ and make $X \times \mathbb{R}$ a metric space using the metric $\rho$ defined by $\rho((x_1, \alpha_1), (x_2, \alpha_2)) = \max \{d(x_1, x_2), |\alpha_1 - \alpha_2|\}$. Since, in $X \times \mathbb{R}$, closed and $\rho$-bounded sets are compact, the parallel body of each closed set will again be a closed set, a fact which we shall often use in the sequel. If $h$ and $g$ are in $UC(X)$ let $d_1(h, g) = \sup_{x \in X} |h(x) - g(x)|$. If $h$ is u.s.c., then the graph of $h$ will not in general be a closed set. Denote its closure in $X \times \mathbb{R}$ by $\overline{h}$. If $h$ and $g$ are in $UC(X)$ denote the Hausdorff distance from $\overline{h}$ to $\overline{g}$ by $d_2(h, g)$. Although $d_2(h, g) = \infty$ is a possibility, $d_2$ is an extended valued metric, not just a pseudometric. Next, if $h$ is u.s.c. its hypograph $\{(x, \alpha) : \alpha \leq h(x)\}$ will be symbolized by $\text{hypo } h$. It is well known that u.s.c. functions are characterized as those having closed hypographs. If $h$ and $g$ are in $UC(X)$ denote the Hausdorff distance between their hypographs by $d_3(h, g)$. Surprisingly, $d_3$ is a finite valued metric on $UC(X)$ even though $h$ and $g$ may be unbounded.

We shall need the following facts established in [2] which we state as theorems.

**Theorem A.** If $h$ and $g$ are in $UC(X)$ then $d_1(h, g) \geq d_2(h, g) \geq d_3(h, g)$.  

**THEOREM B.** The metrics $d_1$ and $d_2$, when restricted to $C(X)$, define the same topology.

**THEOREM C.** Let $\{h_n\}$ be a sequence of u.s.c. functions on $X$. Then $\{h_n\}$ converges to $h$ in the metric $d_3$ if and only if

1. for each $x$ in $X$ whenever $\{x_n\} \to x$ then $\limsup_{n \to \infty} h_n(x_n) \leq h(x)$,
2. for each $x$ in $X$ there exists a sequence $\{x_n\}$ convergent to $x$ for which $\lim_{n \to \infty} h_n(x_n) = h(x)$.

In view of Theorem C above, first proved by Mosco for convex functions [8], $d_3$-convergence is equivalent to the notion of *infimal convergence* for lower semicontinuous functions introduced by Wijisman [11] (but only in the context of compact spaces). Infimal convergence was then studied in a more general setting by DeGiorgetti and Franzoni [4], and has received recent attention from Dolecki, Salinetti, and Wets [5] where it is now called *epi-convergence*.

We consider the metric $d_3$ first. Since this metric is in some sense a Baire category dual to the metric $d_1$ [1], one expects that the Dini class induced by $d_3$ would also be all of $UC(X)$. This conjecture is correct. We need only verify condition (3) in the definition. To this end let $\{h_n\}$ be a decreasing sequence of u.s.c. functions convergent pointwise to an u.s.c. function $h$. Fix $x$ in $X$ and let $\epsilon$ be positive. Choose $N$ so large that $h_N(x) < h(x) + \frac{\epsilon}{2}$ and $\delta > 0$ such that if $d(y, x) < \delta$ then $h_N(y) < h_N(x) + \frac{\epsilon}{2}$. Clearly if $d(y, x) < \delta$ and $n \geq N$ we have $h_n(y) < h(x) + \epsilon$. This implies that whenever $\{x_n\} \to x$ we obtain $\limsup_{n \to \infty} h_n(x_n) < h(x) + \epsilon$. Condition (1) of Theorem C now follows. Condition (2) of Theorem C is already satisfied, for $\lim_{n \to \infty} h_n(x) = h(x)$.

Since $d_1$ and $d_2$, when restricted to $C(X)$, are topologically equivalent, one would suppose that they induce the same Dini class. This is not the case: $d_2$ induces a larger one than $C(X)$ provided that $X$ is locally connected. To properly describe this class we need some preliminary material.

Let $h$ be a real valued function on $X$, not necessarily u.s.c. For each $x \in X$ let

$$L(h, x) = \{a : a \in \mathbb{R} \text{ and } \exists \{x_n\} \to x \text{ for which } \{h(x_n)\} \to a\}.$$ 

Notice that $h(x)$ is an element of $L(h, x)$; it is the largest (resp. smallest) element of $L(h, x)$ if $h$ is u.s.c. (resp. l.s.c.) at $x$. For each $n \in \mathbb{Z}^+$ let

$$L_n(h, x) = \text{cl}\{h(y) : d(x, y) \leq \frac{1}{n}\} = \text{cl} h(B_1/n[x]).$$

We leave the proof of the following simple lemma to the reader.

**LEMMA 1.** Let $X$ be a compact metric space and let $h: X \to \mathbb{R}$. For each $x \in X$, $\bigcap_{n=1}^{\infty} L_n(h, x) = \{a : (x, a) \in \overline{h}\} = L(h, x)$.

As a result of Lemma 1, $L(h, x)$ is a closed set. It is possible that each set $L(h, x)$ can be bounded without $h$ itself being bounded, e.g., define $h: [0, 1] \to \mathbb{R}$ by $h(0) = 0$ and $h(x) = \frac{1}{x}$ otherwise.

**LEMMA 2.** Let $X$ be a compact metric space and let $h: X \to \mathbb{R}$. Suppose for each $x \in X$ the set $L(h, x)$ is bounded. The following are equivalent:

(i) $h$ is bounded.
(ii) For each $x$, $\{L_n(h, x)\}$ converges to $L(h, x)$ in the Hausdorff metric.
(iii) The multifunction $\Gamma$ on $X$ defined by $\Gamma(x) = L(h, x)$ is upper semicontinuous.
(i)\(\rightarrow\)(ii) Fix \(x\) in \(X\). Since for each \(n\), \(L_n(h, x) \supseteq L(h, x)\), we need only show that given \(\epsilon > 0\) the parallel body \(B_\epsilon[L(h, x)]\) contains \(\{L_n(h, x)\}\) eventually. If not, since \(B_\epsilon[L(h, x)]\) is closed and \(\{L_n(h, x)\}\) is a decreasing sequence of sets, we can find, for each \(n\), a point \(x_n\) satisfying \(d(x_n, x) < \frac{1}{n}\) but \(h(x_n) \notin B_\epsilon[L(h, x)]\). By hypothesis \(\{h(x_n)\}\) is bounded; so, it has a convergent subsequence. The limit cannot belong to \(L(h, x)\), for \(L(h, x) \subset \text{int}B_\epsilon[L(h, x)]\). This violates the definition of \(L(h, x)\).

(ii)\(\rightarrow\)(iii) Let \(x\) be arbitrary and let \(V\) be an open set in \(R\) containing \(\Gamma(x)\). We must produce a neighborhood \(U\) of \(x\) such that if \(y \in U\) then \(\Gamma(y) \subset V\). Since \(\Gamma(x) = L(h, x)\) is compact, there exists \(\epsilon > 0\) such that \(B_\epsilon[L(h, x)]\) \(\subset V\). Choose \(n\) so large that \(L_n(h, x) \subset B_\epsilon[L(h, x)]\). If \(d(x, y) < \frac{1}{n}\) and \(\frac{1}{m} < \frac{1}{n} - d(x, y)\) then

\[\Gamma(y) = L(h, y) \subset L_m(h, y) \subset L_n(h, x) \subset B_\epsilon[L(h, x)] \subset V.\]

(iii)\(\rightarrow\)(i) Suppose \(h\) fails to be bounded. Since \(X\) is compact we can find a convergent sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} |h(x_n)| = \infty\). Set \(x = \lim_{n \to \infty} x_n\). Clearly, \(\{h(x_n)\}\) cannot be in the bounded set \(B_\epsilon[L(h, x)]\) eventually. Since \(B_\epsilon[L(h, x)]\) is a neighborhood of \(L(h, x)\) and for each \(n\), \(h(x_n) \in \Gamma(x_n)\), the multifunction \(\Gamma\) fails to be u.s.c. at \(x\).

Next let \(\Omega = \{h: h\ is\ u.s.c.\ and\ bounded\ and\ \forall x, L(h, x)\ is\ convex\}\). Notice that the inclusion \(\Omega \supset C(X)\) is proper. For example \(f: [0, 1] \to R\) defined by

\[f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}\]

is in \(\Omega \supset C(X)\). If \(h \in \Omega\) then each set \(L(h, x)\) is precisely the closed line segment \([ \lim \inf_{y \to x} h(y) , h(x) ]\). Finally, \(\Omega\) may be described as the class of u.s.c. functions for which the multifunction \(\Gamma\) of Lemma 2 is both compact convex valued and u.s.c.

**Lemma 3.** \(\Omega = \{h: h\ is\ bounded\ and\ u.s.c.\ and\ \overline{h} \supset \text{hypo } h - \text{int}(\text{hypo } h)\}\).

**Proof.** Let \(h\) be bounded and u.s.c. It is easy to see that \((x, \alpha) \in \text{hypo } h - \text{int}(\text{hypo } h)\) if and only if \(\lim \inf_{y \to x} h(y) \leq \alpha \leq h(x)\). If \(h\) is u.s.c. then \(\text{hypo } h \subset \overline{h}\), then for each \(x\), Lemma 1 implies that the segment \([ \lim \inf_{y \to x} h(y) , h(x) ]\) \(\subset L(h, x)\). Since the other inclusion always holds whenever \(h\) is u.s.c., we obtain \(h \in \Omega\). Conversely, if \(h \in \Omega\) then for each \(x\), \(L(h, x) = [ \lim \inf_{y \to x} h(y) , h(x) ]\) so that if \((x, \alpha)\) lies in \(\text{hypo } h - \text{int}(\text{hypo } h)\), it must also lie in \(\overline{h}\) by Lemma 1.

Local connectivity of \(X\) is not required to establish condition (3) for \(\Omega\) to be a \(d_2\)-induced Dini class of functions. Before we can prove this we need one more lemma.

**Lemma 4.** Let \(X\) be a compact metric space and let \(\{h_n\}\) be a sequence in \(UC(X)\) \(d_2\)-convergent to a bounded u.s.c. function \(h\). Then for each \(\lambda > 0\) there exists \(N \in \mathbb{Z}^+\) such that \(n \geq N\) implies \(\overline{h} \subset B_{\lambda} [\overline{h_n}]\).

**Proof.** If not, there exists \(\lambda\) and a subsequence of \(\{h_n\}\) such that \(\overline{h}\) fails to lie in the \(\lambda\)-parallel body of the closure of the graph of each term of the subsequence. By relabeling we can assume that, for each \(n\), \(\overline{h} \not\subset B_{\lambda} [\overline{h_n}]\). Since \(B_{\lambda} [\overline{h_n}]\) is a closed set, for each \(n\) we can find \((x_n, h(x_n))\) in \(\overline{h} - B_{\lambda} [\overline{h_n}]\). Once again passing to a subsequence we can assume that \(\{(x_n, h(x_n))\}\) is convergent and is therefore Cauchy. Choose \(N\) so large that \(m > n \geq N\) implies that \(\rho((x_n, h(x_n)), (x_m, h(x_m))) < \frac{\lambda}{2}\).

By the second condition in Theorem C there exists \(K \geq N\) and a point in \(\overline{h}_K\) that
is within $\frac{1}{2}$ of $(x_N, h(x_N))$. It follows that $(x_K, h(x_K))$ has distance at most $\lambda$ from $h_K$, a contradiction.

We note that if $\{h_n\}$ is a sequence of bounded u.s.c. functions $d_3$-convergent to $h$, then $\{h_n\}$ “half-$d_2$ converges” to $h$ in the above sense if and only if $h$ is bounded.

**THEOREM 1.** Let $X$ be a compact metric space with metric $d$. Let $h \in \Omega$ and let $\{h_n\}$ be a decreasing sequence of u.s.c. functions convergent pointwise to $h$. Then $\{h_n\}$ $d_2$-converges to $h$.

**PROOF.** Let $\lambda$ be positive and set $E_\lambda = \bigcup_{x \in X} B_\lambda\{(x, \alpha) : \alpha \in L(h, x)\}$. We claim that if $\epsilon$ is sufficiently small then $B_\epsilon[hypo h] - hypo h \subseteq E_\lambda$. If not we can find for each $\alpha \in Z^+$ a point $(x_n, \alpha_n)$ such that $\alpha_n > h(x_n)$, $(x_n, \alpha_n) \in B_{1/n}[hypo h]$, but $(x_n, \alpha_n) \notin E_\lambda$. By passing to a subsequence we can assume $(x_n)$ converges to some point $x$. Since hypo $h$ is a closed set and $\bigcap_{n=1}^{\infty} B_{1/n}[hypo h] = hypo h$, it is easy to see that $\limsup_{n \to \infty} \alpha_n \leq h(x)$. Set $\alpha = \limsup_{n \to \infty} \alpha_n$. Since $h \in \Omega$ and

$$\alpha \geq \liminf_{n \to \infty} h(x_n) \geq \liminf_{y \to x} h(y)$$

we have $\alpha \in L(h, x)$. There exists $n \in Z^+$ such that $d(x_n, x) \leq \lambda$ and $|\alpha_n - \alpha| \leq \lambda$. However, this implies $(x_n, \alpha_n) \in E_\lambda$, a contradiction. This establishes the claim.

Next, choose a fixed $\epsilon$ for which $B_\epsilon[hypo h] - hypo h \subseteq E_\lambda$. Since $\{h_n\}$ $d_3$-converges to $h$, there exists $N \in Z^+$ such that $n \geq N$ implies hypo $h_n \subseteq B_\epsilon[hypo h]$. As a consequence since $B_\epsilon[hypo h]$ is closed, $\overline{h_n} \subseteq B_\epsilon[hypo h]$ for each such $n$. Since each $h_n$ majorizes $h$, for all $x$, either $(x, h_n(x)) \in \overline{h}$ or $(x, h_n(x)) \in B_\epsilon[hypo h] - hypo h$. In the first case $(x, h_n(x))$ has distance zero from $h$. In the second case, by the definition of $E_\lambda$, $(x, h_n(x))$ has distance at most $\lambda$ from some point $(x, \alpha)$ where $\alpha \in L(h, x)$. By Lemma 1, $(x, \alpha) \in \overline{h}$; so $(x, h_n(x))$ has distance at most $\lambda$ from $\overline{h}$. Since $x$ was arbitrary, it follows that, for each $n \geq N$, $\overline{h_n} \subseteq B_\lambda[\overline{h}]$ and $\overline{h_n} \subseteq \overline{h}$ holds for all $n$ sufficiently large by Lemma 4.

The next theorem, which shows that $\Omega$ is $d_2$-closed, completes the demonstration that $\Omega$ is a Dini class of functions induced by $d_2$. As the next example shows, the hypothesis that $X$ be locally connected cannot be omitted.

**EXAMPLE.** Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \}$ considered as a subspace of the line. Let $h = \chi_{\{0\}}$ and for each $n$ let $h_n = \chi_{\{0\}} \cup \{x : x \geq n\}$. Then $\{h_n\}$ is a decreasing sequence of continuous functions $d_2$-convergent to $h$. However, since $L(h, 0) = \{0\}$, a nonconvex set, $h$ fails to be in $\Omega$.

**THEOREM 2.** Let $X$ be a compact locally connected metric space. Let $\{h_n\}$ be a sequence in $\Omega$, $d_2$-convergent to an u.s.c. function $h$. Then $h \in \Omega$.

**PROOF.** By the remark following Lemma 4, $h$ is bounded. Fix $x$ in $X$. If $h$ is continuous at $x$, then $L(h, x)$ is convex. Otherwise, choose $\alpha$ strictly between $\liminf_{y \to x} h(y)$ and $h(x)$. By Lemma 1 it suffices to show that $(x, \alpha) \in \overline{h}$. Choose $\epsilon < \alpha - \liminf_{y \to x} h(y)$ and let $C$ be a connected neighborhood of $x$ in $B_{\epsilon/2}[x]$. Next select $\delta > 0$ such that $B_\delta[x] \subseteq C$. There exists $N \in Z^+$ such that $n \geq N$ implies $d_2(h_n, h) < \delta$. By Theorem A, $d_3(h_n, h) \leq d_2(h_n, h)$; so, for each $n \geq N$ we have $B_\delta[(x, \alpha)] \cap hypo h_n \neq \emptyset$. Observing that $B_\delta[(x, \alpha)] \subseteq C \times [\alpha - \delta, \alpha + \delta]$ we distinguish two mutually exclusive cases:

1. $\exists n \geq N$ for which $(C \times [\alpha - \delta, \alpha + \delta]) \cap (\text{hypo } h_n - \text{int(hypo } h_n)) \neq \emptyset$.
2. $\forall n \geq N (C \times [\alpha - \delta, \alpha + \delta]) \cap \text{hypo } h_n \subset \text{int(hypo } h_n)$.
Case (1). For such an integer $n$ select $(y, \beta)$ in $(C \times [\alpha - \delta, \alpha + \delta]) \cap \text{hyp}(h_n - \text{int}(\text{hyp}(hn)))$. Since $h_n \in \Omega$, Lemma 3 says that $(y, \beta) \in \overline{h}_n$. Since $d_2(h, h_n) \leq \frac{\delta}{2}$ there exists $(w, \theta) \in \overline{h}$ with $\rho$-distance from $(y, \beta)$ at most $\frac{\delta}{2}$. It follows that $\rho((w, \theta), (x, \alpha)) \leq \epsilon$. Thus, $B_\epsilon((x, \alpha))$ meets $\overline{h}$, and since $\overline{h}$ is a closed set, we conclude that $(x, \alpha) \in \overline{h}$.

Case (2). By the choice of $N$, for each $n \geq N$, we can find $x_n$ such that $\rho((x_n, h_n(x_n)), (x, h(x))) \leq \delta$. Clearly $h_n(x_n) \geq \alpha - \delta$; so, $(x_n, \alpha - \delta)$ lies in hypo $h_n$. By assumption, for each such $n$ we have

$$\{(w, \alpha - \delta) : w \in C \} \cap \text{hyp}(h_n - \text{int}(\text{hyp}(hn))) = \emptyset.$$  

Thus the relatively open subsets of $C$,

$$E_n = \{ w : w \in C \text{ and } (w, \alpha - \delta) \in \text{int} \text{(hyp}(h_n)) \},$$

$$F_n = \{ w : w \in C \text{ and } (w, \alpha - \delta) \notin \text{hyp}(h_n) \},$$

partition the connected set $C$. Since for each $n \geq N$, $x_n \in E_n$, we conclude that $E_n = C$ for all such $n$. As a result, for each $w \in C$ and each such $n$, we have $h_n(w) > \alpha - \delta \geq \alpha - \frac{\delta}{2} > \liminf_{y \to x} h(y) + \frac{\delta}{2}$. Choose $z \in X$ for which $\rho((z, h(x)), (z, \liminf_{y \to x} h(y))) \leq \frac{\delta}{2}$. By Theorem C, if $n \geq N$ is chosen sufficiently large, then we can find $(w, h_n(w))$ whose $\rho$-distance from $(z, h(x))$ is at most $\frac{\delta}{2}$. It follows that $w \in C$ and $h_n(w) \leq \liminf_{y \to x} h(y) + \frac{\delta}{2}$. This is a contradiction so that Case (2) cannot occur. Hence, we revert to Case (1) and the theorem is proved.

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