PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 1, September 1982

ON DINI'S THEOREM AND A METRIC ON $C(X)$ TOPOLOGICALLY EQUIVALENT TO THE UNIFORM METRIC

GERALD BEER

ABSTRACT. Let X be a compact metric space and let $UC(X)$ denote the u.s.c. real valued functions on X. Let τ be a topology on $UC(X)$. $\Omega \subset UC(X)$ is called a Dini class of functions induced by τ if (1) Ω is τ-closed, (2) $C(X) \subset \Omega$, (3) for each $h \in \Omega$ whenever $\{h_n\}$ is a decreasing sequence of u.s.c. functions convergent pointwise to h, then $\{h_n\}$ τ-converges to h. By Dini's theorem the topology of uniform convergence on $UC(X)$ induces $C(X)$ as its Dini class of functions. As a main result, when X is locally connected we show that the hyperspace topology on $UC(X)$ obtained by identifying each u.s.c. function with the closure of its graph induces a larger Dini class of functions than $C(X)$, even though the restriction of this topology to $C(X)$ agrees with the topology of uniform convergence.

Let X be a compact metric space and let $C(X)$ and $UC(X)$ denote the continuous and upper semicontinuous real valued functions on X, respectively. If $f \in C(X)$ and $\{h_n\}$ is a decreasing sequence of u.s.c. functions on X convergent pointwise to f, then Dini's theorem [**10**] says that the convergence must be uniform. Moreover, if $\{f_n\}$ is a sequence in $C(X)$ convergent uniformly to an u.s.c. function h, then h must be in $C(X)$. Thus, the class $C(X)$, viewed as a subclass of $UC(X)$, is induced by the topology of uniform convergence on $UC(X)$ in the following sense.

DEFINITION. Let X be a compact metric space and let $\Omega \subset UC(X)$. If τ is a topology on $UC(X)$, then Ω is called a *Dini class of functions induced by* τ if

(1) Ω is τ-closed,

(2) $C(X) \subset \Omega$,

(3) for each $h \in \Omega$ whenever $\{h_n\}$ is a decreasing sequence of u.s.c. functions convergent pointwise to h, then $\{h_n\}$ τ-converges to h.

It follows from the definition that τ can induce at most one Dini class. To see this, suppose that Ω and Ω' are both Dini classes of τ. Let $f \in \Omega$ be arbitrary. Since each u.s.c. function on X is a pointwise limit of a decreasing sequence of continuous functions [**6**],

$$f \in \mathrm{cl}_\tau\, C(X) \subset \mathrm{cl}_\tau\, \Omega' = \Omega'.$$

Standard topologies on $UC(X)$ seem to induce nice Dini classes of functions. For example, let μ be a regular Borel measure on X. We first consider the convergence in measure topology on $UC(X)$. Recall that $\{h_n\}$ is said to *converge in measure* to

Received by the editors June 4, 1981 and, in revised form, November 11, 1981.
1980 *Mathematics Subject Classification*. Primary 26A15, 54B20, 54C35.
Key words and phrases. Semicontinuous function, Dini's theorem, Hausdorff metric .

h if for each $\epsilon > 0$ there exists $N \in Z^+$ such that for all $n \geq N$,

$$\mu\{x\colon |h_n(x) - h(x)| > \epsilon\} < \epsilon.$$

It is well known [7] that convergence in this sense is described by a pseudometric on the μ-measurable functions on X:

$$\delta_1(f,g) = \int_X \frac{|f-g|}{1+|f-g|}\, d\mu.$$

The δ_1-topology induces $UC(X)$ as its Dini class. Conditions (1) and (2) hold trivially and condition (3) follows from Egoroff's theorem. In the same context define the $L_1(\mu)$ pseudometric $\delta_2\colon UC(X) \times UC(X) \to [0,\infty]$ by

$$\delta_2(f,g) = \int_X |f-g|\, d\mu.$$

We claim that δ_2 induces $L_1(\mu) \cap UC(X)$, the u.s.c. functions that are μ-integrable, as its Dini class. Condition (1) holds because $L_1(\mu)$ is closed under addition, condition (2) holds because X is compact, and condition (3) follows from the Lebesgue dominated convergence theorem using as a majorant $g\colon X \to [0,\infty)$ defined by

$$g(z) = \max\left\{\max_{x\in X} h_1(x), -\min\{h(z),0\}\right\}.$$

In this paper we shall focus our attention on two hyperspace topologies on $UC(X)$ considered in [2]. In short we identify members of $UC(X)$ with certain closed subsets of $X \times R$ and apply the Hausdorff metric. We review the essential features of this construction. Let Y be a metric space with metric ρ and for each y in Y let $B_\lambda[y]$ denote the closed ball of radius λ about y. If K is a closed subset of Y, the *λ-parallel body* of K, denoted by $B_\lambda[K]$, is the set $\bigcup_{y\in K} B_\lambda[y]$. If C and K are closed sets in Y we define the *Hausdorff distance* of C from K to be

$$D(C,K) = \inf\{\lambda\colon B_\lambda[K] \supset C \text{ and } B_\lambda[C] \supset K\}.$$

We refer the reader to Castaing and Valadier [3] or Nadler [9] for further information. Now let X be a metric space with metric d and make $X \times R$ a metric space using the metric ρ defined by $\rho[(x_1,\alpha_1),(x_2,\alpha_2)] = \max\{d(x_1,x_2), |\alpha_1 - \alpha_2|\}$. Since, in $X \times R$, closed and ρ-bounded sets are compact, the parallel body of each closed set will again be a closed set, a fact which we shall often use in the sequel. If h and g are in $UC(X)$ let $d_1(h,g) = \sup_{x\in X} |h(x) - g(x)|$. If h is u.s.c., then the graph of h will not in general be a closed set. Denote its closure in $X \times R$ by $\overline{h}$. If h and g are in $UC(X)$ denote the Hausdorff distance from $\overline{h}$ to $\overline{g}$ by $d_2(h,g)$. Although $d_2(h,g) = \infty$ is a possibility, d_2 is an extended valued metric, not just a pseudometric. Next, if h is u.s.c. its *hypograph* $\{(x,\alpha)\colon \alpha \leq h(x)\}$ will be symbolized by $\operatorname{hypo} h$. It is well known that u.s.c. functions are characterized as those having closed hypographs. If h and g are in $UC(X)$ denote the Hausdorff distance between their hypographs by $d_3(h,g)$. Surprisingly, d_3 is a finite valued metric on $UC(X)$ even though h and g may be unbounded.

We shall need the following facts established in [2] which we state as theorems.

THEOREM A. *If h and g are in $UC(X)$ then $d_1(h,g) \geq d_2(h,g) \geq d_3(h,g)$.*

THEOREM B. *The metrics d_1 and d_2, when restricted to $C(X)$, define the same topology.*

THEOREM C. *Let $\{h_n\}$ be a sequence of u.s.c. functions on X. Then $\{h_n\}$ converges to h in the metric d_3 if and only if*

(1) *for each x in X whenever $\{x_n\} \to x$ then $\limsup_{n\to\infty} h_n(x_n) \leq h(x)$,*

(2) *for each x in X there exists a sequence $\{x_n\}$ convergent to x for which $\lim_{n\to\infty} h_n(x_n) = h(x)$.*

In view of Theorem C above, first proved by Mosco for convex functions [8], d_3-convergence is equivalent to the notion of *infimal convergence* for lower semicontinuous functions introduced by Wijsman [11] (but only in the context of compact spaces). Infimal convergence was then studied in a more general setting by DeGiorgi and Franzoni [4], and has received recent attention from Dolecki, Salinetti, and Wets [5] where it is now called *epi-convergence.*

We consider the metric d_3 first. Since this metric is in some sense a Baire category dual to the metric δ_1 [1], one expects that the Dini class induced by d_3 would also be all of $UC(X)$. This conjecture is correct. We need only verify condition (3) in the definition. To this end let $\{h_n\}$ be a decreasing sequence of u.s.c. functions convergent pointwise to an u.s.c. function h. Fix x in X and let ϵ be positive. Choose N so large that $h_N(x) < h(x) + \frac{\epsilon}{2}$ and $\delta > 0$ such that if $d(y,x) < \delta$ then $h_N(y) < h_N(x) + \frac{\epsilon}{2}$. Clearly if $d(y,x) < \delta$ and $n \geq N$ we have $h_n(y) < h(x) + \epsilon$. This implies that whenever $\{x_n\} \to x$ we obtain $\limsup_{n\to\infty} h_n(x_n) < h(x) + \epsilon$. Condition (1) of Theorem C now follows. Condition (2) of Theorem C is already satisfied, for $\lim_{n\to\infty} h_n(x) = h(x)$.

Since d_1 and d_2, when restricted to $C(X)$, are topologically equivalent, one would suppose that they induce the same Dini class. This is not the case: d_2 induces a larger one than $C(X)$ provided that X is locally connected. To properly describe this class we need some preliminary material.

Let h be a real valued function on X, not necessarily u.s.c. For each $x \in X$ let

$$L(h,x) = \{\alpha\colon \alpha \in R \text{ and } \exists\{x_n\} \to x \text{ for which } \{h(x_n)\} \to \alpha\}.$$

Notice that $h(x)$ is an element of $L(h,x)$; it is the largest (resp. smallest) element of $L(h,x)$ if h is u.s.c. (resp. l.s.c.) at x. For each $n \in Z^+$ let

$$L_n(h,x) = \mathrm{cl}\{h(y)\colon d(x,y) \leq \tfrac{1}{n}\} = \mathrm{cl}\, h(B_{1/n}[x]).$$

We leave the proof of the following simple lemma to the reader.

LEMMA 1. *Let X be a compact metric space and let $h\colon X \to R$. For each $x \in X$, $\bigcap_{n=1}^{\infty} L_n(h,x) = \{\alpha\colon (x,\alpha) \in \overline{h}\} = L(h,x)$.*

As a result of Lemma 1, $L(h,x)$ is a closed set. It is possible that each set $L(h,x)$ can be bounded without h itself being bounded, e.g., define $h\colon [0,1] \to R$ by $h(0) = 0$ and $h(x) = \frac{1}{x}$ otherwise.

LEMMA 2. *Let X be a compact metric space and let $h\colon X \to R$. Suppose for each $x \in X$ the set $L(h,x)$ is bounded. The following are equivalent:*

(i) *h is bounded.*

(ii) *For each x, $\{L_n(h,x)\}$ converges to $L(h,x)$ in the Hausdorff metric.*

(iii) *The multifunction Γ on X defined by $\Gamma(x) = L(h,x)$ is upper semicontinuous.*

PROOF. (i)→(ii) Fix x in X. Since for each n, $L_n(h,x) \supset L(h,x)$, we need only show that given $\epsilon > 0$ the parallel body $B_\epsilon[L(h,x)]$ contains $\{L_n(h,x)\}$ eventually. If not, since $B_\epsilon[L(h,x)]$ is closed and $\{L_n(h,x)\}$ is a decreasing sequence of sets, we can find, for each n, a point x_n satisfying $d(x_n,x) \leq \frac{1}{n}$ but $h(x_n) \notin B_\epsilon[L(h,x)]$. By hypothesis $\{h(x_n)\}$ is bounded; so, it has a convergent subsequence. The limit cannot belong to $L(h,x)$, for $L(h,x) \subset \operatorname{int} B_\epsilon[L(h,x)]$. This violates the definition of $L(h,x)$.

(ii)→(iii) Let x be arbitrary and let V be an open set in R containing $\Gamma(x)$. We must produce a neighborhood U of x such that if $y \in U$ then $\Gamma(y) \subset V$. Since $\Gamma(x) = L(h,x)$ is compact, there exists $\epsilon > 0$ such that $B_\epsilon[L(h,x)] \subset V$. Choose n so large that $L_n(h,x) \subset B_\epsilon[L(h,x)]$. If $d(x,y) < \frac{1}{n}$ and $\frac{1}{m} < \frac{1}{n} - d(x,y)$ then

$$\Gamma(y) = L(h,y) \subset L_m(h,y) \subset L_n(h,x) \subset B_\epsilon[\Gamma(x)] \subset V.$$

(iii)→(i) Suppose h fails to be bounded. Since X is compact we can find a convergent sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} |h(x_n)| = \infty$. Set $x = \lim_{n\to\infty} x_n$. Clearly, $\{h(x_n)\}$ cannot be in the bounded set $B_\epsilon[L(h,x)]$ eventually. Since $B_\epsilon[L(h,x)]$ is a neighborhood of $L(h,x)$ and for each n, $h(x_n) \in \Gamma(x_n)$, the multifunction Γ fails to be u.s.c. at x.

Next let $\Omega = \{h\colon h \text{ is u.s.c. and bounded and } \forall x,\ L(h,x) \text{ is convex}\}$. Notice that the inclusion $\Omega \supset C(X)$ is proper. For example $f\colon [0,1] \to R$ defined by

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

is in $\Omega - C(X)$. If $h \in \Omega$ then each set $L(h,x)$ is precisely the closed line segment $[\liminf_{y\to x} h(y), h(x)]$. Finally, Ω may be described as the class of u.s.c. functions for which the multifunction Γ of Lemma 2 is both compact convex valued and u.s.c.

LEMMA 3. *$\Omega = \{h\colon h$ is bounded and u.s.c. and $\overline{h} \supset \operatorname{hypo} h - \operatorname{int}(\operatorname{hypo} h)\}$.*

PROOF. Let h be bounded and u.s.c. It is easy to see that $(x,\alpha) \in \operatorname{hypo} h - \operatorname{int}(\operatorname{hypo} h)$ if and only if $\liminf_{y\to x} h(y) \leq \alpha \leq h(x)$. If $\operatorname{hypo} h - \operatorname{int}(\operatorname{hypo} h) \subset \overline{h}$, then for each x, Lemma 1 implies that the segment $[\liminf_{y\to x} h(y), h(x)] \subset L(h,x)$. Since the other inclusion always holds whenever h is u.s.c., we obtain $h \in \Omega$. Conversely, if $h \in \Omega$ then for each x, $L(h,x) = [\liminf_{y\to x} h(y), h(x)]$ so that if (x,α) lies in $\operatorname{hypo} h - \operatorname{int}(\operatorname{hypo} h)$, it must also lie in $\overline{h}$ by Lemma 1.

Local connectivity of X is not required to establish condition (3) for Ω to be a d_2-induced Dini class of functions. Before we can prove this we need one more lemma.

LEMMA 4. *Let X be a compact metric space and let $\{h_n\}$ be a sequence in $UC(X)$ d_3-convergent to a bounded u.s.c. function h. Then for each $\lambda > 0$ there exists $N \in Z^+$ such that $n \geq N$ implies $\overline{h} \subset B_\lambda[\overline{h_n}]$.*

PROOF. If not, there exists λ and a subsequence of $\{h_n\}$ such that $\overline{h}$ fails to lie in the λ-parallel body of the closure of the graph of each term of the subsequence. By relabeling we can assume that, for each n, $\overline{h} \not\subset B_\lambda[\overline{h_n}]$. Since $B_\lambda[\overline{h_n}]$ is a closed set, for each n we can find $(x_n, h(x_n))$ in $\overline{h} - B_\lambda[\overline{h_n}]$. Once again passing to a subsequence we can assume that $\{(x_n, h(x_n))\}$ is convergent and is therefore Cauchy. Choose N so large that $m > n \geq N$ implies that $\rho[(x_n, h(x_n)), (x_m, h(x_m))] < \frac{\lambda}{2}$. By the second condition in Theorem C there exists $K \geq N$ and a point in $\overline{h_K}$ that

is within $\frac{\lambda}{2}$ of $(x_N, h(x_N))$. It follows that $(x_K, h(x_K))$ has distance at most λ from $\overline{h}_K$, a contradiction.

We note that if $\{h_n\}$ is a sequence of bounded u.s.c. functions d_3-convergent to h, then $\{h_n\}$ "half-d_2 converges" to h in the above sense if and only if h is bounded.

THEOREM 1. *Let X be a compact metric space with metric d. Let $h \in \Omega$ and let $\{h_n\}$ be a decreasing sequence of u.s.c. functions convergent pointwise to h. Then $\{h_n\}$ d_2-converges to h.*

PROOF. Let λ be positive and set $E_\lambda = \bigcup_{x \in X} B_\lambda[\{(x, \alpha)\colon \alpha \in L(h, x)\}]$. We claim that if ϵ is sufficiently small then $B_\epsilon[\text{hypo}\, h] - \text{hypo}\, h \subset E_\lambda$. If not we can find for each $n \in Z^+$ a point (x_n, α_n) such that $\alpha_n > h(x_n)$, $(x_n, \alpha_n) \in B_{1/n}[\text{hypo}\, h]$, but $(x_n, \alpha_n) \notin E_\lambda$. By passing to a subsequence we can assume $\{x_n\}$ converges to some point x. Since hypo h is a closed set and $\bigcap_{n=1}^{\infty} B_{1/n}[\text{hypo}\, h] = \text{hypo}\, h$, it is easy to see that $\limsup_{n\to\infty} \alpha_n \leq h(x)$. Set $\alpha = \limsup_{n\to\infty} \alpha_n$. Since $h \in \Omega$ and

$$\alpha \geq \liminf_{n\to\infty} h(x_n) \geq \liminf_{y\to x} h(y)$$

we have $\alpha \in L(h, x)$. There exists $n \in Z^+$ such that $d(x_n, x) \leq \lambda$ and $|\alpha_n - \alpha| \leq \lambda$. However, this implies $(x_n, \alpha_n) \in E_\lambda$, a contradiction. This establishes the claim.

Next, choose a fixed ϵ for which $B_\epsilon[\text{hypo}\, h] - \text{hypo}\, h \subset E_\lambda$. Since $\{h_n\}$ d_3-converges to h, there exists $N \in Z^+$ such that $n \geq N$ implies $\text{hypo}\, h_n \subset B_\epsilon[\text{hypo}\, h]$. As a consequence since $B_\epsilon[\text{hypo}\, h]$ is closed, $\overline{h}_n \subset B_\epsilon[\text{hypo}\, h]$ for each such n. Since each h_n majorizes h, for all x, either $(x, h_n(x)) \in \overline{h}$ or $(x, h_n(x)) \in B_\epsilon[\text{hypo}\, h] - \text{hypo}\, h$. In the first case $(x, h_n(x))$ has distance zero from $\overline{h}$. In the second case, by the definition of E_λ, $(x, h_n(x))$ has distance at most λ from some point (x, α) where $\alpha \in L(h, x)$. By Lemma 1, $(x, \alpha) \in \overline{h}$; so $(x, h_n(x))$ has distance at most λ from $\overline{h}$. Since x was arbitrary, it follows that, for each $n \geq N$, $\overline{h}_n \subset B_\lambda[\overline{h}]$. The inclusion $B_\lambda[\overline{h}_n] \supset \overline{h}$ holds for all n sufficiently large by Lemma 4.

The next theorem, which shows that Ω is d_2-closed, completes the demonstration that Ω is a Dini class of functions induced by d_2. As the next example shows, the hypothesis that X be locally connected cannot be omitted.

EXAMPLE. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ considered as a subspace of the line. Let $h = \chi_{\{0\}}$ and for each n let $h_n = \chi_{\{0\} \cup \{\frac{1}{k}\colon k \geq n\}}$. Then $\{h_n\}$ is a decreasing sequence of continuous functions d_2-convergent to h. However, since $L(h, 0) = \{0, 1\}$, a nonconvex set, h fails to be in Ω.

THEOREM 2. *Let X be a compact locally connected metric space. Let $\{h_n\}$ be a sequence in Ω, d_2-convergent to an u.s.c. function h. Then $h \in \Omega$.*

PROOF. By the remark following Lemma 4, h is bounded. Fix x in X. If h is continuous at x, then $L(h, x)$ is convex. Otherwise, choose α strictly between $\liminf_{y\to x} h(y)$ and $h(x)$. By Lemma 1 it suffices to show that $(x, \alpha) \in \overline{h}$. Choose $\epsilon < \alpha - \liminf_{y\to x} h(y)$ and let C be a connected neighborhood of x in $B_{\epsilon/2}[x]$. Next select $\delta > 0$ such that $B_\delta[x] \subset C$. There exists $N \in Z^+$ such that $n \geq N$ implies $d_2(h_n, h) < \delta$. By Theorem A, $d_3(h_n, h) \leq d_2(h_n, h)$; so, for each $n \geq N$ we have $B_\delta[(x, \alpha)] \cap \text{hypo}\, h_n \neq \emptyset$. Observing that $B_\delta[(x, \alpha)] \subset C \times [\alpha - \delta, \alpha + \delta]$ we distinguish two mutually exclusive cases:

(1) $\exists n \geq N$ for which $(C \times [\alpha - \delta, \alpha + \delta]) \cap (\text{hypo}\, h_n - \text{int}(\text{hypo}\, h_n)) \neq \emptyset$.

(2) $\forall n \geq N$ $(C \times [\alpha - \delta, \alpha + \delta]) \cap \text{hypo}\, h_n \subset \text{int}(\text{hypo}\, h_n)$.

Case (1). For such an integer n select (y, β) in $(C \times [\alpha - \delta, \alpha + \delta]) \cap (\text{hypo}\, h_n - \text{int}(\text{hypo}\, h_n))$. Since $h_n \in \Omega$, Lemma 3 says that $(y, \beta) \in \overline{h}_n$. Since $d_2(h, h_n) \leq \frac{\epsilon}{2}$ there exists $(w, \theta) \in \overline{h}$ with ρ-distance from (y, β) at most $\frac{\epsilon}{2}$. It follows that $\rho[(w, \theta), (x, \alpha)] \leq \epsilon$. Thus, $B_\epsilon[(x, \alpha)]$ meets $\overline{h}$, and since $\overline{h}$ is a closed set, we conclude that $(x, \alpha) \in \overline{h}$.

Case (2). By the choice of N, for each $n \geq N$, we can find x_n such that $\rho[(x_n, h_n(x_n)), (x, h(x))] \leq \delta$. Clearly $h_n(x_n) \geq \alpha - \delta$; so, $(x_n, \alpha - \delta)$ lies in $\text{hypo}\, h_n$. By assumption, for each such n we have

$$\{(w, \alpha - \delta) \colon w \in C\} \cap (\text{hypo}\, h_n - \text{int}(\text{hypo}\, h_n)) = \emptyset.$$

Thus the relatively open subsets of C,

$$E_n = \{w \colon w \in C \text{ and } (w, \alpha - \delta) \in \text{int}(\text{hypo}\, h_n)\},$$
$$F_n = \{w \colon w \in C \text{ and } (w, \alpha - \delta) \notin \text{hypo}\, h_n\},$$

partition the connected set C. Since for each $n \geq N$, $x_n \in E_n$, we conclude that $E_n = C$ for all such n. As a result, for each $w \in C$ and each such n, we have $h_n(w) > \alpha - \delta \geq \alpha - \frac{\epsilon}{2} > \liminf_{y \to x} h(y) + \frac{\epsilon}{2}$. Choose $z \in X$ for which $\rho[(z, h(x)), (x, \liminf_{y \to x} h(y))] \leq \frac{\delta}{2}$. By Theorem C, if $n \geq N$ is chosen sufficiently large, then we can find $(w, h_n(w))$ whose ρ-distance from $(z, h(z))$ is at most $\frac{\delta}{2}$. It follows that $w \in C$ and $h_n(w) \leq \liminf_{y \to x} h(y) + \frac{\epsilon}{2}$. This is a contradiction so that Case (2) cannot occur. Hence, we revert to Case (1) and the theorem is proved.

BIBLIOGRAPHY

1. G. Beer, *A natural topology for upper semicontinuous functions and a Baire category dual for convergence in measure*, Pacific J. Math. **96** (1981), 251–263.
2. ——, *Upper semicontinuous functions and the Stone approximation theorem*, J. Approximation Theory **34** (1982), 1–11.
3. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Springer-Verlag, Berlin and New York, 1977.
4. E. DeGiorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), 842–850.
5. S. Dolecki, G. Salinetti and R. Wets, *Convergence of functions: equi-semicontinuity*, Proc. London Math. Soc. (to appear).
6. R. Engleking, *General topology*, Polish Scientific Publishers, Warsaw, 1977.
7. E. Hewitt and C. Stromberg, *Real and abstract analysis*, Springer-Verlag, Berlin and New York, 1965.
8. U. Mosco, *Convergence of convex sets and the solutions of variational inequalities*, Adv. in Math. **3** (1969), 510–585.
9. S. Nadler, *Hyperspaces of sets*, Dekker, New York, 1978.
10. R. Royden, *Real analysis*, Macmillan, New York, 1968.
11. R. Wijsman, *Convergence of sequences of convex sets, cones, and functions.* II, Trans. Amer. Math. Soc. **123** (1966), 32–45.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, LOS ANGELES, CALIFORNIA 90032