A PATHOLOGICAL AREA PRESERVING 
$C^\infty$ DIFFEOMORPHISM OF THE PLANE

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ABSTRACT. The pseudocircle $P$ is an hereditarily indecomposable planar continuum. In particular, it is connected but nowhere locally connected. We construct a $C^\infty$ area preserving diffeomorphism of the plane with $P$ as a minimal set. The diffeomorphism $f$ is constructed as an explicit limit of diffeomorphisms conjugate to rotations about the origin. There is a well-defined irrational rotation number for $f\mid P$ even though $f\mid P$ is not even semiconjugate to a rotation of $S^1$. If we remove the requirement that our diffeomorphisms be area preserving, we may alter the example so that $P$ is an attracting minimal set.

The complexity of a dynamical system is reflected in part by its invariant sets. We consider here a simple dynamical system, the action of a diffeomorphism on $\mathbb{R}^2$. Pathology abounds in the compact connected subsets of $\mathbb{R}^2$, and we show that this pathology will occur in the minimal sets of diffeomorphisms, even if we restrict ourselves to those which are $C^\infty$ and area preserving.

We choose the pseudocircle $P$ (defined below) as our model of extreme pathology. Its key feature is that it is hereditarily indecomposable. Indecomposable means that $P$ cannot be written as the union of two proper compact connected subsets, and hereditarily indecomposable means that every compact connected subset of $P$ is indecomposable. ($P$ is, for instance, nowhere locally connected.) What makes $P$ tractable in spite of this behavior, is that Bing's construction of pseudocircles [B2] is both simple and malleable. It is an infinite construction allowing choices at each stage. In light of [F], the resulting space is, to a great extent, independent of the choices.

Using an infinite limit, we construct an embedding of $P$ in $\mathbb{R}^2$ and a $C^\infty$ area preserving diffeomorphism $f: \mathbb{R}^2 \to \mathbb{R}^2$ with $P$ as a minimal set. (One may take the domain of $f$ to be an annulus $A^2$.) There are two other features of $f$ that are worth mentioning. $P$ is defined as the intersection of annuli $P = \bigcap_{n=1}^\infty A_n$ where each inclusion $A_{n+1} \to A_n$ is a homotopy equivalence. It therefore makes sense to speak of a rotation number for $f$ and indeed $f$ has a well-defined irrational rotation number. Nonetheless, $f$ is not semiconjugate to a homeomorphism of $S^1$. Second, if one is willing to consider $C^\infty$ diffeomorphisms which do not preserve area, $f$ is easily perturbed to $f': \mathbb{R}^2 \to \mathbb{R}^2$ with $P$ as an attracting minimal set.

It is especially relevant that $f$ is area preserving, as there is a long history of interest in area preserving diffeomorphisms of a surface. In particular, Birkhoff [Bir] (see also the recent work of Mather [M]) studied invariant sets which were the frontiers of invariant, open, simply connected regions, and gave criterion forcing...
the invariant set to be a Jordan curve. To my knowledge, ours is the first such ex-
ample where the area preserving diffeomorphism f acts transitively on the invariant
set P but P is not a Jordan curve.

I would like to thank W. Thurston for pointing out that f is not semiconjugate
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The construction. Begin with $C_1$, the unit circle in $\mathbb{R}^2$, and $\epsilon_1 > 0$. Divide $C_1$
into $p_1$ equal pieces $D_1, \ldots, D_{p_1}$, each of diameter less than $\epsilon_1$, and thicken radially
to $A_1 = D_1 \times [-1,1]$ so that each $a_i = D_i \times [-1,1]$ still has diameter less than $\epsilon_1$.
Define $f_1: \mathbb{R}^2 \to \mathbb{R}^2$ so that $f_1|C_1 \times [-1,1] = R_{\alpha_1} \times \text{identity}$ where $\alpha_1 = 1/p_1$
and $R_{\alpha}$ is rotation through the angle $\alpha$.

Embed a circle $C_2$ in $A_1$ which is crooked with respect to $a_1, \ldots, a_{p_1}$ and invariant
with respect to $f_1$. A more general definition of crookedness is given in [B1] but
for our purposes, the following suffices. The inclusion $C_2 \to A_1$ is a homotopy
equivalence. Consider the universal covers $\tilde{C}_2 \subset \tilde{A}_1$. The division of $A_1$ into $a_i$,
$i = 1, \ldots, p_1$, lifts to a division of $\tilde{A}_1$ into $\tilde{a}_i$, $i = -\infty, \ldots, \infty$. $C_2$ is crookedly
embedded in $A_1$ with respect to $a_1, \ldots, a_{p_1}$ if any segment $\alpha$ of $\tilde{C}_2$ running from
$\tilde{a}_i$ to $\tilde{a}_j$ ($p_1 > j - i > 2$) can be written as a composition $\alpha = \alpha_1 \cdot \alpha_2 \cdot \alpha_3$ where
\(\alpha_1\) runs from \(\tilde{a}_i\) to \(\tilde{a}_{j-1}\), \(\alpha_2\) runs from \(\tilde{a}_{j-1}\) to \(\tilde{a}_{i+1}\), and \(\alpha_3\) runs from \(\tilde{a}_{i+1}\) to \(\tilde{a}_j\).

For our construction, \(C_2\) must be invariant by \(f_1\). Let \(T \cong c \times [-1,1] \subset C_1 \times [-1,1]\) for some \(c \in C_1\), and let \(T \cdot S\) denote the algebraic intersection of \(T\) with an arc \(S\). Choose a crooked arc \(S_1\) (see Figure 3) which spirals down toward the inner boundary component of \(A_1\) and which satisfies \(T \cdot S_1 = p_1 - 2\). (An arc \(S\) in \(C_1 \times [-1,1]\) is crooked if any segment \(a\) with \(T \cdot a = k > 2\) can be decomposed into 3 subarcs \(a = a_1 \cdot a_2 \cdot a_3\) such that \(T \cdot a_1 = T \cdot a_3 = -T \cdot a_2 + 1 = k - 1\).) Push all but the first loop of \(S_1\) slightly above itself obtaining a crooked arc \(S_2\) spiraling toward the outer boundary of \(A_1\) and satisfying \(T \cdot S_2 = p_1 - 3\). Connect the endpoints of \(S_1\) to those of \(S_2\) without intersecting \(T\). This gives a degree one circle \(C'_2\) in \(A_1\) whose \(p_1\)-fold cover \(C_2\) is both crooked with respect to \(\alpha_1, \ldots, \alpha_{p_1}\) and invariant by \(f_1 = R_{\alpha_1} \times \text{id}\).

We return to the construction of \(f\) and \(P\). Choose \(\epsilon_2 > 0\) and a rational number \(\alpha_2 = \alpha_1 + 1/m_2\) where \(m_2\) is a large positive integer (conditions on how large \(m_2\) must be, will be enumerated later). Since \(C_2\) is smoothly embedded in \(C_1 \times [-1,1]\) and \(f_1|C_1 \times [-1,1] = R_{\alpha_1} \times \text{id}\), \(C_2\) can be thickened to an annulus \(A_2 = C_2 \times [-1/2,1/2] \subset C_2 \times [-1,1] \subset C_1 \times [-1,1]\) (this last inclusion does not map fibers into fibers) so that \(f_1|C_2 \times [-1,1] = R_{\alpha_1} \times \text{id}\) and so that \(C_2 \times [-1,1]\), with the metric induced from \(C_1 \times [-1,1]\), is isometric to \(S^1 \times [-\epsilon'_2/4,\epsilon'_2/4]\), where \(S^1\) is the circle of length \(l\) (= length of \(C_2\)) in \(R^2\) and \(\epsilon'_2 < \epsilon_2\).

Define \(f_2 : R^2 \to R^2\) by

\[
\begin{align*}
f_2 = \begin{cases}
f_1 \text{ on } R^2 - (C_2 \times [-1,1]), \\
R_{\alpha_2} \times \text{id} \text{ on } C_2 \times [-1/2,1/2], \\
R_{\Phi([1/2,1])} \times \text{id} \text{ on } C_2 \times ([-1,-1/2] \cup [1/2,1]),
\end{cases}
\end{align*}
\]

where \(\Phi : [1/2,1] \to [\alpha_2, \alpha_1]\) is smooth and is constant on neighborhoods of \(\{1/2\}\) and \(\{1\}\).

Note that \(f_2\) is area preserving and that if \(m_1\) is sufficiently large, then \(f_2\) is \(\epsilon_2\)-close to \(f_1\) in the \(C^2\) topology. Let \(p_2\) be the period of \(R_{\alpha_2}\) and partition \(A_2\) into \(p_2\) pieces \(b_1, \ldots, b_{p_2}\) transitively permuted by \(f_2\).

This completes a cycle in the inductive construction of \(f_n\) and \(A_n\). To construct \(f_3\) and \(A_3\), repeat the steps above: embed \(C_3\) crookedly in \(A_2\) with respect to
b₁,...,bₚ₃, and invariantly with respect to f₂; thicken C₃ to A₃ and let f₃|A₃ be
f₂ followed by a slight rotation of A₃ in its own annular structure, not that of A₂;
partition A₃ with respect to f₃.

Iterate this to construct fₙ, Aₙ and εₙ with the following additional properties:

1. fₙ: R² → R² is an area preserving C∞ diffeomorphism which is εₙ-close to
fₙ₋₁ in the Cn topology.

2. Aₙ is partitioned into pₙ elements of diameter less than εₙ. These elements
are transitively permuted by fₙ.

3. The εₙ's are so small that f = limₙ→∞ fₙ is a C∞ area preserving diffeomor-
phism and |fᵢ(x) − fᵢ(₁)(x)| < εₙ for 0 ≤ i ≤ pₙ and all x ∈ Aₙ.

Conditions (2) and (3) guarantee that f acts minimally on P = ∩ₙ₌₁ Aₙ which,
by [F], is the pseudocircle.

It remains to verify the two additional features of f mentioned in the intro-
duction. It is easy to construct a C∞ diffeomorphism h: S¹ × [0,1) → A₁ — P such that
each h(S¹ × {t}) is invariant by f. Perturbing f slightly in this induced structure
we easily construct a C∞ diffeomorphism f': R² → R² with P as an attracting
minimal set.

Showing that f|P is not semiconjugate to a homeomorphism of S¹ is more
involved and we will only outline the argument. Suppose then that there exists
a homeomorphism g: S¹ → S¹ and an onto map Π: P → S¹ such that

\[
\begin{array}{ccc}
P & \xrightarrow{f} & P \\ \downarrow \Pi & & \downarrow \Pi \\ S¹ & \xrightarrow{g} & S¹
\end{array}
\]

commutes.

Let γₙ: Aₙ → S¹ be the projection of the annulus to its core circle and let
\(\tilde{γ}_n\): \(\tilde{A}_n\) → R be the lift of γₙ to the corresponding universal covers. The inclusions
Aₙ₊₁ ⊂ Aₙ are homotopy equivalences, so \(\tilde{A}_n₊₁ \subset \tilde{A}_n\) and \(\tilde{P} = \cap_{n=1}^{∞} \tilde{A}_n\) is an
infinite cyclic cover of $P$. The commutative diagram above lifts to another com-
mutative diagram.

\[
\begin{array}{ccc}
\tilde{P} & \xrightarrow{\tilde{f}} & \tilde{P} \\
\tilde{n} \downarrow & & \downarrow \tilde{n} \\
R & \xrightarrow{\tilde{g}} & R
\end{array}
\]

It follows that $\tilde{\Pi}$ is proper (if the image of $\tilde{\Pi}$ were compact, then $g$ would have a
fixed point and $\Pi$ would not map $P$ onto $S^1$) and hence that: $\forall n > 0 \exists B_n$ such that
$|\tilde{\gamma}_n(\tilde{x}) - \tilde{\gamma}_n(\tilde{y})| < 1 \Rightarrow |\tilde{\gamma}_n(\tilde{f}_n^k(\tilde{x})) - \tilde{\gamma}_n(\tilde{f}_n^k(\tilde{y}))| < B_n$ for $\forall k$ and $\forall \tilde{x}, \tilde{y} \in \tilde{P}$.

In contrast to this, consider the way that a fundamental domain of $\tilde{A}_m$ is
spreading out in $\tilde{A}_n$ for $m > n$. It is easy to check that if the partitioning of
$A_k$ is sufficiently fine for all $k$, then a fundamental domain of $\tilde{A}_m$ will intersect
$m - n + 1$ fundamental domains of $\tilde{A}_n$. In Figure 5 the crookedness of $\tilde{A}_m$ has
been surpressed.

\[\text{FIGURE 5}\]

Let $\tilde{a}_i$, $i = -\infty, \ldots, \infty$, be the partition of $\tilde{A}_m$. Choose $i_1 < i_2 < i_3$ so that
(i) $\tilde{\gamma}_n(\tilde{a}_{i_1}), \tilde{\gamma}_n(\tilde{a}_{i_3}) \subset [(m - n)/2, (m - n + 1)/2],$
(ii) $\tilde{\gamma}_n(\tilde{a}_{i_2}) \subset [0, 1],$
(iii) $\tilde{\gamma}_n(\tilde{a}_i) > (m - n)/2$ for $i > i_3$.

Since $f_m$ permutes the $p_m$ elements of the partition of $A_m$, there exist $1 \leq k \leq p_m$
and $j \in \mathbb{Z}$ such that $t^j \cdot f_m^k(\tilde{a}_{i_1}) = \tilde{a}_{i_3}$ where $t$ generates the deck transformations
of $\tilde{A}_m$. Thus for any $\tilde{x} \in \tilde{a}_{i_1}$ and $\tilde{y} \in \tilde{a}_{i_3}$,
$|\tilde{\gamma}_n(\tilde{x}) - \tilde{\gamma}_n(\tilde{y})| < 1$ and $|\tilde{\gamma}_n(\tilde{f}_m^k(\tilde{x})) - \tilde{\gamma}_n(\tilde{f}_m^k(\tilde{y}))| > (m - n)/2 - 1$. If the $\varepsilon_i$’s are sufficiently small, then the last equation
will hold with $f$ replacing $\tilde{f}_m$. This contradiction completes the proof that $f: P \to
P$ is not semiconjugate to a homeomorphism of $S^1$.

Finally, we indicate why $f|P$ has a well-defined rotation number. It follows from
the construction of $P$ that $|\tilde{\gamma}_n - \tilde{\gamma}_1| < 2n$. To show that $\alpha = \lim_{n \to \infty} \alpha_n$ is the
rotation number of $f|P$, we must show that

$$\lim_{k \to \infty} \frac{\tilde{\gamma}_1(\tilde{f}_n^k(\tilde{x})) - \tilde{\gamma}_1(\tilde{x})}{k} = \alpha \quad \text{for all } \tilde{x} \in \tilde{P}.$$
Fix \( k > 0 \) and choose \( n \) so that \( 2^{n-1} \leq k < 2^n \). By definition, \( \gamma_n j_n^k(x) - \tilde{\gamma}_n(x) = k\alpha_n \). If the \( \epsilon_i \)'s are sufficiently small, then \( |\gamma_n j_n^k - \tilde{\gamma}_n|^k < 1 \) for \( 0 \leq k \leq 2^n \). Thus
\[
\left| \frac{\gamma_n j_n^k(x) - \tilde{\gamma}_n(x)}{k} - \alpha \right| \leq |\alpha - \alpha_n| + \frac{4n}{k} + \frac{1}{k},
\]
which completes the proof.

REFERENCES


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