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A NONMETRIC INDECOMPOSABLE HOMOGENEOUS CONTINUUM EVERY PROPER SUBCONTINUUM OF WHICH IS AN ARC

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ABSTRACT. We construct a nonmetric indecomposable homogeneous continuum with the property that each of its proper subcontinua is an arc.

In [5] the second author showed that in the metric case, homogeneous indecomposable continua having only arcs as proper subcontinua are solenoids. In [4] the first author defined a continuum that is indecomposable, nonmetrizable, having only arcs for proper subcontinua. Here we show that this continuum is homogeneous. This answers a question that the first author asked at the Warsaw Geometric Topology Conference in 1978.

Let us start by redefining the example. Let $S = \{e^{it}: t$ is a real number and $i = \sqrt{-1}\}$ denote the unit circle in the complex plane. If $a = e^{ir}$ and $b = e^{is}$, then $[a, b)$ denotes the set $\{c \in S: c = e^{it}$ and $r \leq t < s\}$, and $(a, b] = ([a, b) \cup \{b\}) \backslash \{a\}$. Let $\mathcal{D} = S \times \{0, 1\}$ and let the topology on $\mathcal{D}$ be defined by pairs of half-open intervals $[a, b) \times \{1\} \cup (a, b] \times \{0\}$, where $a, b \in S$. The space $\mathcal{D}$ is sometimes called the double arrow space. Let $f: \mathcal{D} \to \mathcal{D}$ be a homeomorphism defined by $f\langle e^{it}, j\rangle = \langle e^{it+i}, j\rangle$. The quotient space K obtained from the space $\mathcal{D} \times [0, 1]$ by identification of points $\langle d, 0\rangle$ with $\langle f(d), 1\rangle$, for each $d \in \mathcal{D}$, is, as shown in [4], an indecomposable continuum. The same result can be obtained using theorems proved in [3]. Let q_f be the quotient map from $\mathcal{D} \times [0, 1]$ onto K.

We will show that every proper subcontinuum of K is an arc. This argument also will give us another proof that K is indecomposable. Let M be a proper subcontinuum of K. Let $U = ([a, b) \times \{1\} \times (u, v)) \cup ((a, b] \times \{0\} \times (u, v))$ be an open set in $K \backslash M$ such that $0 < u < v < 1$ and for some point $q_f\langle p, r\rangle$ of M, p does not belong to $[a, b] \times \{0, 1\}$. Assume without loss of generality that $p \in S \times \{1\}$. Let m be the smallest positive integer such that $f^m(p) \in [a, b) \times \{1\}$. Let n be the smallest positive integer such that $f^{-n}(p) \in [a, b) \times \{1\}$. Let L be the arc

$$q_f\Big(\{f^{-n}(p)\} \times [0, u] \\ \cup \bigcup \{\{f^k(p)\} \times [0, 1]: k = -n + 1, \ldots, m - 1\} \cup \{f^m(p)\} \times [v, 1]\Big)$$

in $K \backslash U$ from $q_f\langle f^{-n}(p), u\rangle$ to $q_f\langle f^m(p), v\rangle$.

Next we show that M is a subarc of L. To accomplish this let z be a point of $K \backslash (L \cup U)$. There is a closed-open set D in $\mathcal{D}$ such that $p \in D$ and $f^{-n}(D) \cup$

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$f^m(D) \subseteq [a,b) \times \{1\} \cup (a,b] \times \{0\}$, and z does not belong to

$$E = q_f\Big(f^{-n}(D) \times [0,u] \cup \bigcup\{f^k(D) \times [0,1]\colon k = -n+1,\dots,m-1\} \cup f^m(D) \times [v,1]\Big).$$

The set E is a closed-open subset of $K\backslash U$ that contains L and misses z. Since M is a connected subset of $K\backslash U$, it follows that M is a subset of L.

A continuum is indecomposable if and only if it does not have a proper subcontinuum with nonvoid interior. Since every proper subcontinuum of K is an arc and no arc in K contains an open subset of K, it follows that K is indecomposable.

To show that K is homogeneous, we consider two families of composants. One, say $\mathcal{A}$, consists of composants passing through points $q_f(x)$, where $x \in (\mathcal{S} \times \{0\} \times \{0\})$ and the second, say $\mathcal{B}$, consists of composants passing through points $q_f(y)$, where $y \in (\mathcal{S} \times \{1\} \times \{0\})$. Clearly, $\bigcup \mathcal{A} \cap \bigcup \mathcal{B} = \emptyset$ and $\bigcup \mathcal{A} \cup \bigcup \mathcal{B} = K$.

Suppose that $p, q \in \bigcup \mathcal{A}$. Then $p = q_f\langle\langle e^{is}, 0\rangle, t\rangle$ and $q = q_f\langle\langle e^{ir}, 0\rangle, \tau\rangle$, where $0 \le t < 1$ and $0 \le \tau < 1$. Define a mapping f_1 from K onto itself by letting

$$f_1(q_f\langle\langle e^{iv}, j\rangle, w\rangle) = q_f\langle\langle e^{i(v+r-s)}, j\rangle, w\rangle.$$

The mapping f_1 is a homeomorphism. Define f_2 from K onto itself by letting

$$f_2(q_f\langle\langle e^{iv}, j\rangle, w\rangle) = \begin{cases} q_f\langle\langle e^{iv+i}, j\rangle, w+\tau-t-1\rangle & \text{if } w+\tau-t \ge 1, \\ q_f\langle\langle e^{iv}, j\rangle, w+\tau-t\rangle & \text{if } 0 \le w+\tau-t < 1, \\ q_f\langle\langle e^{iv-i}, j\rangle, w+\tau-t+1\rangle & \text{if } w+\tau-t < 0. \end{cases}$$

The mapping f_2 is a homeomorphism and $f_2 \circ f_1$ is a homeomorphism that takes p to q. The same holds for $p, q \in \bigcup \mathcal{B}$.

Suppose that $p \in \bigcup \mathcal{A}$ and $q \in \bigcup \mathcal{B}$. Define a homeomorphism $\overline{f}_3$ from $\mathcal{D} \times [0,1]$ onto itself by letting $\overline{f}_3\langle\langle e^{iv}, j\rangle, w\rangle = \langle\langle e^{-iv}, 1-j\rangle, 1-w\rangle$. Observe that

$$\overline{f}_3\langle\langle e^{iv}, j\rangle, 0\rangle = \langle\langle e^{-iv}, 1-j\rangle, 1\rangle$$

and

$$\overline{f}_3\langle\langle e^{iv+i}, j\rangle, 1\rangle = \langle\langle e^{-(iv+i)}, 1-j\rangle, 0\rangle.$$

Hence $\overline{f}_3$ induces a homeomorphism f_3 from K onto itself. This homeomorphism takes a point of $\bigcup \mathcal{A}$ to a point of $\bigcup \mathcal{B}$. In particular, $f_3(p) \in \bigcup \mathcal{B}$. Now, as in the preceding case, we define a homeomorphism that takes p to q.

Note that the preceding arguments hold for a general class of continua. To show that K is an indecomposable homogeneous Hausdorff continuum every nondegenerate proper subcontinuum of which is an arc, we need only that $\mathcal{D}$ is a compact, totally disconnected space such that there is a homeomorphism f of $\mathcal{D}$ onto $\mathcal{D}$ with each orbit dense and for each pair x, y of points of $\mathcal{D}$ there is a homeomorphism g of $\mathcal{D}$ onto $\mathcal{D}$ with $g(x) = y$ such that g respects the orbits of f. Hence it is possible to construct more varied examples than just that from the double arrow space $\mathcal{D}$.

Question 1. Can every homogeneous Hausdorff continuum every nondegenerate proper subcontinuum of which is an arc be obtained from a construction of this type?

The example K (described above) is not circularly chainable. Thus we ask the following:

Question 2. Does there exist a circularly chainable nonmetrizable homogeneous indecomposable continuum having only arcs for nondegenerate proper subcontinua?

For more information and questions about homogeneous Hausdorff continua see [2].

Let X be a compact and totally disconnected Hausdorff space and let f be a homeomorphism from X onto itself. Let $X \times I/f$ denote the space obtained from the product $X \times I$ by identifying points $\langle x, 0\rangle$ and $\langle f(x), 1\rangle$. We show that covering dimension of $X \times I/f$ is one.

Let q_f denote the quotient mapping from $X \times I$ onto $X \times I/f$. Since it contains an arc, its dimension is greater than zero. Let $\mathcal{U}$ be an open cover of $X \times I/f$. Because $X \times I/f$ is compact, we can assume that $\mathcal{U}$ is finite. The base of $X \times I/f$ consists of sets of the form $q_f(D \times (r, s))$ and $q_f(D \times [0, r) \cup f(D) \times (s, 1])$, where D is a closed-open subset of X and $0 < r < s < 1$. Hence we can assume that $\mathcal{U}$ consists of such sets only, say $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, where

$$\mathcal{U}_1 = \{q_f(D_n \times (r_n, s_n)) \colon n = 1, \dots, k_0\}$$

and

$$\mathcal{U}_2 = \{q_f(D_n \times [0, r_n) \cup f(D_n) \times (s_n, 1]) \colon n = k_0 + 1, \dots, k\}.$$

And more, $|s_n - r_n| < \frac{1}{3}$ for $n = 1, \dots, k_0$, and $r_n < \frac{1}{3}$ and $\frac{2}{3} < s_n$ for $n = k_0 + 1, \dots, k$.

There is a refinement $\mathcal{V}_1$ of $\mathcal{U}_1$ such that $\bigcup \mathcal{V}_1 = \bigcup \mathcal{U}_1$ and for each $q_f(E_1 \times (r, s))$, $q_f(E_2 \times (u, v)) \in \mathcal{V}_1$ we have $E_1 \cap E_2 = \emptyset$ or $E_1 = E_2$. If $E_1 = E_2$ then $r < u < s < v$ or $u < r < v < s$. Furthermore, any three sets belonging to $\mathcal{V}_1$ have no points in common.

There is a refinement $\mathcal{V}_2$ of $\mathcal{U}_2$ consisting of disjoint sets. Let

$$W = \bigcup\{V \in \mathcal{U}_1 \colon V \text{ intersects two other sets of } \mathcal{V}_1\}.$$

Let $\mathcal{V}_3 = \{V \setminus \operatorname{cl} W \colon V \in \mathcal{V}_2\}$. The family $\mathcal{V}_1 \cup \mathcal{V}_3$ is an open cover of $X \times I/f$ that refines $\mathcal{U}$ such that the intersection of any three sets is empty. Hence the covering dimension of $X \times I/f$ is one.

It follows from this general argument that our example K has covering dimension one.

In [1] H. Cohen proved that for compact spaces the cohomological dimension is not greater than the covering dimension. J. D. Newburgh [7, Theorem 2.9, p. 290] proved that every compact connected group with finite cohomological dimension is a separable metric space. Hence the continuum K is not a topological group. Furthermore, it is not a semigroup with identity. As proved in [6] a finite dimensional clan (i.e. compact connected semigroup with identity) is a group.

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