PROPER PSEUDOCOMPACT EXTENSIONS OF COMPACT ABELIAN GROUP TOPOLOGIES

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ABSTRACT. A compact Abelian group $G$ admits a strictly finer pseudocompact topological group topology if and only if the weight of $G$ is uncountable.

1. Introduction and motivation. Many authors have considered questions concerning increasing or reducing the topology of a topological group without disturbing certain specific features. For example: Can the topology of a locally compact group be enlarged to a locally compact group without augmenting the family of closed subgroups [12, 11, 10]? Does every (Hausdorff) group, if not totally bounded, admit a coarser Hausdorff topological group topology [11, 14, 6]? Does every totally bounded topological group topology extend to a strictly larger one [2]?

The present paper is a contribution in this vein. Motivated by questions in topological Galois theory [13, 15], where pseudocompact groups play a basic rôle [16], and by a fascination with pseudocompact groups as a subject of interest in its own right [3, 20, 21, 1], we show that every compact Abelian group with uncountable weight does admit a strictly finer pseudocompact group topology. Our point of departure is [4], where the question we consider was raised, and answered for totally disconnected groups.

The referee has pointed out that there is a close connection between our result and Theorem 4.4 of Hewitt and Ross [8]. A sharpening of the technique of [8] can be used to prove a result much stronger than that of the present paper, and to demonstrate the existence of finer pseudocompact group topologies of (maximal) weight $2^{2^{\omega}}$ when $G$ is compact Abelian with $w(G) = \alpha > \omega$. We intend to discuss this in another paper.

In an attempt to make this work reasonably self-contained, we record from [4] those arguments necessary for a complete proof of our theorem.

2. Definitions; Results from the literature. All the topological groups hypothesized in this paper are assumed to satisfy the $T_0$ separation property; as is well known (see for example [7, Theorem 8.4]), this guarantees that they are in fact completely regular, Hausdorff spaces, i.e., Tychonoff spaces.

A topological group $G$ is said to be totally bounded if for every nonempty, open subset $U$ of $G$ there is finite $F \subseteq G$ such that $G = FU$. It is a theorem of Weil...
that every totally bounded group \( G \) embeds as a dense subgroup of a compact group \( \hat{G} \); further, \( \hat{G} \) is unique in the sense that if \( \hat{G} \) is a compact group in which \( G \) is dense then there is a function \( \varphi \), simultaneously an algebraic isomorphism and a topological homeomorphism, from \( G \) onto \( \hat{G} \), such that \( \varphi(x) = x \) for all \( x \in G \). The group \( \hat{G} \) is called the Weil completion of the (totally bounded) group \( G \).

A topological space \( X \) is said to be pseudocompact if every continuous, real-valued function on \( X \) has bounded range; for Tychonoff spaces this condition is equivalent to the condition that every locally finite family of nonempty open subsets of \( X \) is finite. It is not difficult to show that every pseudocompact group \( G \) is totally bounded \([3]\) (and hence is dense in the compact group \( \hat{G} \)). The following result from \([3]\) helps to explain which totally bounded groups are pseudocompact.

2.1. Let \( K \) be a compact group and \( G \) a dense subgroup of \( K \). Then \( G \) is pseudocompact if and only if \( G \) is \( G_\delta \)-dense in \( K \) in the sense that every nonempty \( G_\delta \) of \( K \) has nonempty intersection with \( G \).

2.2. LEMMA. Let \( G \) and \( \hat{G} \) be compact groups, \( \varphi \) a continuous homomorphism of \( G \) onto \( \hat{G} \), and \( E \) a nonempty \( G_\delta \) in \( G \). Then \( \varphi \) is an open function, and \( \varphi[E] \) contains a nonempty (compact) \( G_\delta \) in \( \hat{G} \).

PROOF. That \( \varphi \) is open follows from the Baire category theorem. The set \( E \) contains a set of the form \( \bigcap_n U_n \), with \( \cap G U_{n+1} \subset U_n \) and \( U_n \) open in \( G \) for \( n < \omega \). The set \( \bigcap_n \varphi[U_n] \), which is \( \bigcap_n \varphi[\cap G U_n] \), is then as required.

We remark in passing that (as is shown in \([4]\)) it follows without difficulty from (2.2) that if \( G \) and \( \hat{G} \) are compact groups, \( \varphi \) a continuous homomorphism from \( G \) onto \( \hat{G} \) and \( H \) a dense, pseudocompact subgroup of \( \hat{G} \), then \( \varphi^{-1}(H) \) is a dense, pseudocompact subgroup of \( G \). In certain contexts this observation makes it easy to define dense, pseudocompact subgroups of a given compact group \( G \). Since the result is not needed here, we omit the details of the proof.

In addition to the usual elements of Pontrjagin duality, as set forth, for example, in \([7]\), we need the following algebraic results concerning the \( p \)-group decomposition of an Abelian torsion group.

We denote by \( P \) the set of prime numbers.

2.3. (See \([7, A.3]\)). Let \( G \) be an Abelian torsion group and for \( p \in P \) let \( G_p \) be the set of elements of \( G \) whose order is a power of \( p \). Then \( G_p \) is a \( p \)-primary subgroup of \( G \), and \( G \) is isomorphic to the group \( \bigoplus_{p \in P} G_p \).

2.4. (See \([7, A.11(III)]\)). Let \( G \) be an Abelian \( p \)-group with socle \( S = \{x \in G : \text{order of } x = p\} \cup \{e\} \), for \( x \in G \) let \( p^{k(x)} \) be the order of \( x \) in \( G \) and let \( L \) be a maximal independent set in \( G \). Then \( \{p^{k(x)} - x : x \in L\} \) is a basis for \( S \).

2.5. COROLLARY. Let \( G \) be an Abelian \( p \)-group with \( |G| > \omega \) and let \( S \) be the socle of \( G \). Then \( |S| = |G| \).

PROOF. With \( L \) as in 2.4 we have \( |G| \leq |L| \cdot \omega \) and \( |S| \geq |L| \); from \( |G| > \omega \) follows \( |S| = |G| \), as required.

For a space \( X \) and \( x \in X \), the symbols \( w(X) \) and \( \chi(x,X) \) denote respectively the weight of \( X \) and the local weight of \( X \) at \( x \).

The following statements can be substantially generalized, but they are adequate to our needs in their present form.

2.6. (See \([7, \text{Theorem 24.15 and 9, 28.58(c)}]\). Let \( G \) be a compact Abelian group with dual group \( \hat{G} \). Then \( \chi(e,G) = w(G) = |G| \).
3. Proof of the theorem. This first, elementary lemma is a special case of a general topological result noticed by R. M. Stephenson, Jr. [17]; the present proof is taken from [4].

3.1. Lemma. Let \( G = (G, \mathcal{T}) \) be a compact group with \( w(G) \leq \omega \) and let \( \mathcal{T}' \) be a pseudocompact group topology for \( G \) such that \( \mathcal{T}' \supset \mathcal{T} \). Then \( \mathcal{T}' = \mathcal{T} \).

Proof. Each point of \( G \) is a \( \mathcal{T}' - G \) s of \( G \) and hence of the Weil completion \( \tilde{G} \) of \( (G, \mathcal{T}') \). It follows that every point of \( \tilde{G} \) is a \( G_r \). Thus \( \chi(p, \tilde{G}) \leq \omega \) for all \( p \in \tilde{G} \) and hence \( \tilde{G} \) is metrizable (cf. [7, Theorem 8.3]). Thus \( (G, \mathcal{T}') \) is a pseudocompact metric space, hence is compact (see for example [5, Exercise 3D.2]). The identity function from \( (G, \mathcal{T}') \) onto \( (G, \mathcal{T}) \) is then a homeomorphism, as required.

For topological groups \( G \) and \( \tilde{G} \), we write \( G \cong \tilde{G} \) if some function from \( G \) onto \( \tilde{G} \) is both an isomorphism and a topological homeomorphism.

In what follows we denote by \( \mathbb{Z} \) and \( T \) the set of integers and the circle, each endowed with the usual algebraic and topological properties. For \( p \in P \) the (additive) group of integers mod \( p \) is denoted \( \mathbb{Z}(p) \), and for \( 0 < n < \omega \) we write \( T(n) = \{ z \in T : \text{in} = 1 \} \).

Definition. For a cardinal number \( \alpha \), a subgroup \( K \) of \( T^\alpha \) is a \( T \)-elementary subgroup of \( I^\alpha \) if there is a closed subgroup \( S \) of \( T \), with \( |S| > 1 \), such that \( K = S^\alpha \).

We note that \( T^\alpha \) is itself \( T \)-elementary in \( TQ \). Indeed, the \( T \)-elementary subgroups of \( T^\alpha \) are exactly \( T^\alpha \) and \( T(n)^\alpha \) for \( 0 < n < \omega \).

3.2. Lemma. Let \( G \) be a compact, Abelian group with \( w(G) > \omega \). Then

(a) there is a subgroup \( H \) of \( \tilde{G} \) such that either \( H \cong \bigoplus_{i \in I} \mathbb{Z}_i \) with \( |I| > \omega \) or \( H \cong \bigoplus_{i \in I} \mathbb{Z}(p)_i \) (for some \( p \in P \)) with \( |I| > \omega \); and

(b) there exist \( \alpha > \omega \) and a \( T \)-elementary subgroup \( K \) of \( T^\alpha \) and a continuous homomorphism of \( G \) onto \( K \).

Proof. (a) Let \( \tau_0 \) and \( \tau_p \) (\( p \in P \)) denote respectively the torsion-free rank and the \( p \)-rank, and let \( M \) denote the injective hull (that is, the minimal divisible extension) of \( \tilde{G} \). Using (2.6) and [7, Theorems A.14 and A.16] we have

\[
\omega < w(G) = |\tilde{G}| \leq |M| = \omega \left[ \tau_0(M) + \sum_{p \in P} \tau_p(M) \right]
\]

it follows that either \( \tau_0(\tilde{G}) > \omega \) or there is \( p \in P \) such that \( \tau_p(\tilde{G}) > \omega \). In the first case we take

\[
H = \bigoplus_{i \in I} \mathbb{Z}_i \quad \text{with} \quad |I| = \tau_0(\tilde{G});
\]

and in the second case, with \( H \) the socle of \( \tilde{G}_p \), we have

\[
\tilde{G} \supset H \cong \bigoplus_{i \in I} \mathbb{Z}(p)_i \quad \text{with} \quad |I| = \tau_p(\tilde{G}).
\]

(b) The annihilator \( A \) of \( H \) in \( G \), defined by the rule

\[
A = \{ x \in G : \chi \in H \implies \chi(x) = 1 \},
\]
satisfies \( G / A \cong \hat{H} \) (see [7, Theorem 24.11]). When \( H \cong \bigoplus_{i \in I} \mathbb{Z}_i \) we set
\[
K = T' = \prod_{i \in I} (\mathbb{Z}_i)^* = \hat{H},
\]
and when \( H \cong \bigoplus_{i \in I} \mathbb{Z}(p)_i \) we set
\[
K = T(p)' = \prod_{i \in I} (\mathbb{Z}(p)_i)^* = \hat{H};
\]
the canonical homomorphism: \( G \to G / A \cong \hat{H} = K \) is then as required.

3.3. **Lemma.** Let \( G \) be a compact, Abelian group and suppose there is a continuous homomorphism \( \varphi \) of \( G \) onto a \( T \)-elementary subgroup \( K = S^\alpha \subset T^\alpha \) with \( \alpha > \omega \). Then there is homomorphism \( \Omega \) of \( G \) onto \( S \) such that the graph of \( \Omega \) is a dense, pseudocompact subgroup of \( G \times S \).

**Proof.** We set \( T\xi = T \) for \( \xi < \alpha \), and
\[
W = \bigoplus_{\xi < \alpha} T\xi \subset T^\alpha.
\]
We define a homomorphism \( \psi \) from \( W \) to \( T \) by the rule \( \psi(t) = \prod_{\xi < \alpha} t_\xi \) (this is a finite product for each \( t \in W \)) and, appealing to the divisibility of \( T \), we choose any homomorphism \( \tilde{\psi} : T^\alpha \to T \) such that \( \tilde{\psi}|W = \psi \); we define \( \psi = \tilde{\psi}|K \).

It is clear that \( \tilde{\psi}[K] \supset S \)—indeed, \( \tilde{\psi}[K \cap W] \supset S \). We claim \( \tilde{\psi}[K] = S \). If \( S = T \) this is clear, and if \( S = T(n) \) then for \( t \in K \) we have \( t^n = 1 \in T^\alpha \) and hence \( (\tilde{\psi}(t))^n = 1 \in T \), i.e., \( \tilde{\psi}(t) \in S \).

Now we define \( \Omega = \tilde{\psi} \circ \varphi : G \to S \) and we set \( H = \text{graph } \Omega \subset G \times S \). It is clear that \( H \) is a subgroup of the compact group \( G \times S \). To complete the proof it is, by (2.1), enough to show that \( H \) is \( G_\delta \)-dense in \( G \times S \).

Let \( F \) be a nonempty \( G_\delta \) of \( G \times S \). The set \( F \) contains a set of the form \( E \times \{p\} \) with \( E \) a nonempty \( G_\delta \) of \( G \) and with \( p \in S \), and from (2.2) with \( G \) replaced by \( K \) it follows that there is a nonempty \( G_\delta \) \( D \) of \( K \) such that \( D \subset \varphi[E] \). The set \( D \) contains a set of the form
\[
\{t(C)\} \times S^\alpha \setminus C \subset S^\alpha = K
\]
with \( C \) a countable subset of \( \alpha \) and with \( t(C) \in S^\alpha \).

Now define \( \tilde{\xi} \in K \) by
\[
\tilde{\xi} = t(C) \quad \text{for } \xi \in C
\]
and
\[
\tilde{\xi} = 1 \quad \text{for } \xi \in \alpha \setminus C,
\]
set \( q = \tilde{\psi}(\tilde{\xi}) \in S \), choose \( \eta \in \alpha \setminus C \) and define \( \tilde{s} \in K \) by
\[
\tilde{s} = \tilde{\xi} = t(C) \quad \text{for } \xi \in C
\]
and
\[
\tilde{s} = pq^{-1} \in S \quad \text{for } \xi = \eta
\]
and
\[
\tilde{s} = 1 \in S \quad \text{for } \xi \in \alpha \setminus (C \cup \{\eta\}).
\]
Then
\[
\tilde{\psi}(\tilde{s}) = \psi(\tilde{\xi} \cdot \tilde{\xi}^{-1} \cdot \tilde{s}) = q \cdot \psi(\tilde{\xi} \cdot \tilde{s}) = q \cdot pq^{-1} = p.
\]
Since \( \tilde{s} \in \{t(C)\} \times S^\alpha \setminus C \subset D \subset \varphi[E] \) there is \( \tilde{x} \in E \) such that \( \varphi(\tilde{x}) = \tilde{s} \), and from
\[
\Omega(\tilde{x}) = \tilde{\psi}(\varphi(\tilde{x})) = \tilde{\psi}(\tilde{s}) = p
\]
we have \( \langle \tilde{x}, \Omega(\tilde{x}) \rangle \in H \cap (E \times \{p\}) \subset H \cap F \), as required. The proof is complete.

The theorem stated in the abstract is now easily proved.
3.4 Theorem. Let $G = (G, \tau)$ be a compact Abelian group. There is a pseudocompact group topology $\tau'$ for $G$ with $\tau' \supsetneq \tau$ if and only if $\omega(G) > \omega$.

Proof. The "only if" statement is Lemma 3.1. For the "if" statement we note from Lemmas 3.2 and 3.3 that there is a homomorphism $\Omega$ of $G$ onto a closed subgroup $S$ of $T$ with $|S| > 1$ such that the set $H = \text{graph } \Omega$ is a dense, pseudocompact subgroup of $G \times S$.

Let $\pi$ be the (continuous) projection from $G \times S$ onto $G$ and $\tau'$ the topology on $G$ induced by $\pi|H$, that is, the topology for $G$ defined by the requirement that $\pi|H$ is a homeomorphism of $H$ onto $G$. That $\tau' \supsetneq \tau$ follows from the fact that $\pi$ is continuous (from $H$ onto $(G, \tau)$). The containment is proper because $(G, \tau)$ is compact while $H$, a proper, dense subspace of $G \times S$, is not compact.

The proof is complete.

4. Concluding remarks. We address Lemma 3.3 and its proof, retaining the notation used there.

4.1. The homomorphism $\psi: W \rightarrow T$ given by $\psi(t) = \prod_{\xi < \alpha} t_\xi$ is not continuous. Indeed if $\psi$ were continuous then some extension $\tilde{\psi}$ from $T_\alpha$ to $T$ would be continuous and then $H$, the graph of $\Omega$, would be closed in $G \times S$; this is absurd since $H$ is dense in $G \times S$ and $H \neq G \times S$. Let us note a direct proof that $\psi$ is not continuous. Let $r$ be any element of $S$ such that $r \neq 1$, let $\{\xi(n): n < \omega\}$ be a strictly increasing sequence of predecessors of $\alpha$, and define $t(n) \in W$ by

$$t(n)_\xi = r \quad \text{for } \xi = \xi(n)$$

$$t(n)_\xi = 1 \quad \text{for } \xi < \alpha, \xi \neq \xi(n).$$

The sequence $t(n)$ approaches the element of $W$ whose coordinates are all 1, and

$$\lim_n \psi(t(n)) = \lim_r = r \neq 1 = \psi\left(\lim_n t(n)\right).$$

4.2. The proof of Lemma 3.3 makes it clear that the graph of $\tilde{\psi}$ is $G_\delta$-dense in $T^\alpha \times T$; this condition is equivalent to the statement that if $C$ is a countable subset of $\alpha$ and $t(C) \in T^C$ and $p \in T$, then there is $\bar{s} \in T^\alpha$ such that $\bar{s}_\xi = t(C)_\xi$ for $\xi \in C$ and $\tilde{\psi}(\bar{s}) = p$. In fact, however, a much stronger statement is true: A value for $\psi$ specified in advance is achieved not only at a point of $T^\alpha$ for which countably many coordinates are specified in advance, but indeed at some point of $T^\alpha$ all but one of whose coordinates are specified. The proof is essentially as before: if $\eta < \alpha$ and $t \in T^\alpha \setminus \{\eta\}$ and $p \in T$, and if we define $\bar{t}$ and $\bar{s} \in T^\alpha$ by

$$\bar{t}_\xi = \bar{s}_\xi = t_\xi \quad \text{for } \xi \neq \eta,$$

$$\bar{t}_\eta = 1, \quad \bar{s}_\eta = p \cdot \tilde{\psi}(t)^{-1},$$

then $\tilde{\psi}(\bar{s}) = p$. Algebraically, we have for each $\eta < \alpha$ the isomorphism

$$(\text{graph } \tilde{\psi}) \times T_\eta = T^\alpha \times T.$$

4.3. The fact that $H = \text{graph } \Omega$ is $G_\delta$-dense in $G \times S$ shows (since $\{1\}$ is a $G_\delta$ of $S$) that if $E$ is a nonempty $G_\delta$ of $G$ then $(\text{kernel } \Omega) \cap E \neq \emptyset$. That is, the discontinuous homomorphism $\Omega: G \rightarrow T$ has these properties: Its graph yields a pseudocompact refinement (a covering) of $(G, \tau)$, and its kernel is a dense, pseudocompact subgroup of $(G, \tau)$. 

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4.4. It is known [3] that a topological group is pseudocompact if and only if it is totally bounded and its Weil completion coincides with its Stone-Čech compactification. It follows that for \((G, \mathcal{T})\) as in Theorem 3.4 the Stone-Čech compactification of \((G, \mathcal{T}')\) is isomorphic to \((G, \mathcal{T}) \times S\). When \(G\) is itself a \(T\)-elementary subgroup \(S^\alpha \subseteq \mathcal{T}^\alpha\) with \(\alpha > \omega\), the finer pseudocompact topology \(\mathcal{T}'\) we have constructed for \(G\) then has the property that the Stone-Čech compactification of \((G, \mathcal{T}')\) is topologically isomorphic to \(S^\alpha \times S\) in its usual topology, i.e., to \(G\) itself.

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