SHORTER NOTES

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GAUSS-BONNET THEOREMS FOR NONCOMPACT SURFACES

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The aim of this note is to give short proofs of the following two theorems, due to Cohn-Vossen [3] and Huber [4] respectively.

THEOREM A (GAUSS-BONNET INEQUALITY). Let \( M \) be a finitely connected complete noncompact Riemannian surface with Gaussian curvature \( K \) and area element \( dA \). If \( \int_M K \, dA \) is absolutely integrable, then \( \chi(M) \geq \int_M K \, dA \).

Theorem B. Let \( M \) be a finitely connected complete, finite volume noncompact Riemannian surface with \( \int_M K \, dA \) absolutely integrable. Then

\[
\chi(M) = \int_M K \, dA.
\]

For Theorem A, see also [1].

Such an \( M \) is homeomorphic to a compact surface with \( p \) points deleted. A neighborhood of each point is homeomorphic to \( S^1 \times \mathbb{R}^+ \), and by forming the gradient flow associated to a Morse function on \( M \) [5], the metric on the cusp \( S^1 \times \mathbb{R}^+ \) can be chosen to be of the form \( g_{11}(\theta, t) \, d\theta^2 + g_{22}(\theta, t) \, dt^2 \). Reparametrizing \( \mathbb{R}^+ \) by arclength puts the metric in the form \( g_{11}(\theta, t) \, d\theta^2 + dt^2 \). Since \( M \) is complete, the new parameterization ranges over all of \( \mathbb{R}^+ \).

Let \( M_h = M - \bigcup_{i} (S^1 \times (h, \infty)) \), so \( M_h \) is just \( M \) truncated at height \( h \) up each cusp. Then the Gauss-Bonnet Theorem for surfaces with boundary gives \( \chi(M_h) = \int_M K \, dA + \int_{\partial M_h} \omega_{12} \), where \( \omega_{12} \) is the connection one-form associated to an orthonormal frame on \( M \) [2]. Since \( \chi(M) = \chi(M_h) \), we must show \( \lim_{h \to \infty} \int_{\partial M_h} \omega_{12} \geq 0 \) for Theorem A and \( \lim_{h \to \infty} \int_{\partial M_h} \omega_{12} = 0 \) for Theorem B.

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Picking the orthonormal frame $\theta^1 = \sqrt{g_{11}} \, d\theta$ and $\theta^2 = dt$ gives $\omega_{12} = (d/dt)(\sqrt{g_{11}}) \, d\theta$ via the first structure equation $d\theta^1 = \omega_{12} \wedge \theta^2$. The second structure equation gives $KdA = \Omega_{12} = d\omega_{12} = (d^2/dt^2)(\sqrt{g_{11}}) \, d\theta \, dt$. Since $\int_M K \, dA < \infty$, $\lim_{h \to \infty} \int_{\partial M_h}(d^2/dt^2)(\sqrt{g_{11}}) \, d\theta = 0$ or $\lim_{h \to \infty} \int_{\partial M_h}(d/dt)(\sqrt{g_{11}}) \, d\theta$ is a constant $C$.

For Theorem B, $\int_M \sqrt{g_{11}} \, d\theta \, dt < \infty$ implies $\lim_{h \to \infty} \int_{\partial M_h} \sqrt{g_{11}} \, d\theta = 0$. Now $\lim_{h \to \infty} \int_{\partial M_h} \omega_{12} = \lim_{h \to \infty}(d/dt) \int_{\sqrt{g_{11}}} \, d\theta = C$ forces $C = 0$.

For Theorem A, we need to show $C > 0$. Since $\int_{\partial M_h} \sqrt{g_{11}} \, d\theta \sim C \cdot h + D$ as $h \to \infty$, if $C < 0$ we get $\int_{\partial M_h} \sqrt{g_{11}} \, d\theta < 0$ for each $h > 0$. Since the integrand is positive, this is impossible.

REFERENCES


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