RULED FUNCTION FIELDS

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ABSTRACT. Let $L = L_1(z_1) = L_2(z_2) \supset K$ where $z_i$ is transcendental over $L_i$, and $L_i$ is a finitely generated transcendence degree 1 extension of $K$, $i = 1, 2$. If the genus of $L_1/K = 0$, then $L_1$ and $L_2$ are $K$-isomorphic. If the genus of $L_1/K > 0$, then $L_1 = L_2$ and moreover $L_1$ is invariant under all automorphisms of $L/K$. A criterion is also established for a subfield of a ruled field $L$ to be ruled.

Let $L$ be a finitely generated extension of a field $K$. $L$ is said to be ruled over $K$ if there exists an intermediate field $L_1$ and an element $x_1$ transcendental over $L_1$, such that $L = L_1(x_1)$. The Zariski problem [6] asks: If $L = L_1(z_1) = L_2(z_2)$ is ruled in two ways over $K$, must $L_1$ and $L_2$ be $K$-isomorphic? The answers to some special cases of the 1-dimensional problem were announced in [6] and here we provide a complete affirmative answer for the 1-dimensional case. Henceforth, we assume the transcendence degree of $L$ over $K$ is 2. If $L_1$ is an intermediate field of $L/K$ of transcendence degree 1 over $K$, the genus of $L_1/K$ is by definition the genus of $L_1$ over the algebraic closure of $K$ in $L_1$.

The proof of the one dimensional case is achieved by examining the possibilities for $L$ to be ruled over two distinct subfields $L_1$ and $L_2$. If $L_1 \cap L_2 = K$, then $L_1$ and $L_2$ must be of genus 0. This leads to the result that if $L = L_1(z_1) = L_2(z_2)$, then $L_1$ and $L_2$ are $K$-isomorphic genus 0 extensions of $K$.

Proposition 1. Suppose $L = L_1(z_1) = L_2(z_2) \supset K$ where $z_i$ is transcendental over $L_i$, and $L_i$ is a finitely generated transcendence degree 1 extension of $K$, $i = 1, 2$. If $L_1 \cap L_2 = K$, then $L_1$ and $L_2$ are $K$-isomorphic genus 0 extensions of $K$.

Proof. Since each $L_i$ is algebraically closed in $L$, the algebraic closure of $K$ in $L$ is contained in each $L_i$. Thus $K$ is algebraically closed in $L$ since $L_1 \cap L_2 = K$. By [4, Theorem 1.1, p. 1304], there exists a unique minimal intermediate field $L^*$ over which $L$ is separable. Since $L$ is separable over $L_1$ and $L_2$, $L^* \subseteq L_1 \cap L_2$. Thus $L^* = K$, i.e., $L$ is separable over $K$. Thus each $L_i$ is separable, hence regular, over $K$. Since $L_1 \cap L_2 = K$, we have $L_1 \nsubseteq L_2$; and therefore some element of $L_1$ is transcendental over $L_2$. Since the transcendence degree of $L_1/K$ is 1, a transcendence basis for $L_1/K$ remains independent over $L_2$, i.e., $L_1$ and $L_2$ are free over $K$. By [5, Theorem 3, p. 57], $L_1$ and $L_2$ are linearly disjoint over $K$. Now, $L_2(x_2) \supset L_1 \supset L_2$, and hence by Luroth's theorem, $L_2L_1$ is simple transcendental.
over $L_2$. Thus $L_2L_1$ is of genus 0 over $L_2$. By [3, Theorem 2, p. 132], $L_1/K$ is of genus 0. By a symmetric argument, $L_2/K$ is also of genus 0.

Recall that a genus 0 extension $L_1$ of a finite field $K$ is pure transcendental. One sees this as follows: since $L_1$ has a divisor of degree 1 [3, Theorem, p. 148] and since the genus is 0, the corollaries to the Riemann-Roch theorem [3, p. 40] show this divisor must be integral, hence a prime divisor of degree 1, and hence $L_1/K$ is simple transcendental [3, Theorem, p. 50]. Thus if $K$ is finite, $L_1$ and $L_2$ are simple transcendental extensions of $K$, and hence are isomorphic.

If $K$ is infinite, [7, Lemma 1, p. 209] shows $L_1$ is $K$-isomorphic to a subfield of $L_2$, and $L_2$ is $K$-isomorphic to a subfield of $L_1$. If $L_1$ is simple transcendental over $K$, then so is $L_2$ by Luroth's theorem. If $L_1$ is not simple transcendental over $K$, then [1, Corollary 11.3, p. 42] shows $L_1$ and $L_2$ are $K$-isomorphic. Q.E.D.

It should be noted that if $L_1$ is a nonrational genus 0 function field over $K$ (char $K^* 
eq 2$) and $L_2$ is a $K$-isomorphic copy of $L_1$, then $L_2L_1$, the free join of $L_1$ and $L_2$, will be ruled over both $L_1$ and $L_2$ [1, p. 41].

**Theorem 2.** Suppose $L = L_1(x_1) = L_2(x_2) \supseteq K$ where $x_i$ is transcendental over $L_i$, and $L_i$ is a finitely generated transcendence degree 1 extension of $K$, $i = 1, 2$. Then $L_1$ and $L_2$ are $K$-isomorphic.

**Proof.** It suffices to show they are isomorphic over their intersection, which contains $K$. If their intersection is algebraic over $K$, then Proposition 1 applies. If it is not algebraic over $K$, then each of $L_1$ and $L_2$ must be the algebraic closure in $L$ of their intersection. Thus they are equal in this case.

**Theorem 3.** Suppose $L = L_1(x_1) \supseteq L_2(x_2) \supseteq K$ where $x_1$ is transcendental over $L_1$ and $L_1$ is a finitely generated transcendence degree 1 extension of $K$. Assume the genus of $L_1/K > 0$. Then $L_1$ is invariant under any $K$-automorphism of $L$.

**Proof.** Let $\alpha$ be a $K$-automorphism of $L$. Then $L = L_1(x_1) = L_2(\alpha(x_1))$. Since $L_1/K$ is not of genus 0, Proposition 1 shows $L_1 \cap L_2^{\alpha}$ cannot be algebraic over $K$. But then $L_1$ and $L_2^{\alpha}$ are both the algebraic closure of $L_1 \cap L_2^{\alpha}$ in $L$, i.e., $L_1 = L_2^{\alpha}$. Q.E.D.

If $L$ is ruled over $K$, must an intermediate field $F$ with $[L : F] < \infty$ also be ruled over $K$? If $K$ is algebraically closed of char 0, [2, Proposition 2, p. 106] shows the answer is yes. For $K$ not algebraically closed (but still of char 0), the answer is no. An example is given in [8, p. 330]. There, $K = C(\mu)$, $L = C(\mu, v, w)$ where $\{\mu, v, w\}$ is algebraically independent over $C$. A subfield $F$ with $[C(\mu, v, w) : F] = 2$ is constructed with $F$ not ruled over $C(\mu)$. Actually, [2] shows $F$ is not pure transcendental over $C(\mu)$. However, if $F$ were ruled, then $F$ would be pure transcendental by the generalized Luroth theorem [6]. However, we can use the results of this paper to get an affirmative answer in some cases.

**Theorem 4.** Let $L = L_1(x_1) \supseteq L_2 \supseteq K$ where $L_1$ is a finitely generated extension of $K$ of transcendence degree 1 and positive genus with $x_1$ transcendental over $L_1$. Let $G$ be a finite group of $K$-automorphisms of $L$ and let $F$ be its fixed field. If $|G|$ is odd, then $F$ is also ruled over $K$.

**Proof.** Since $L_1$ is invariant under the action of $G$ by Theorem 3, it follows from [8, Theorem 4, p. 322] that $F$ is pure transcendental over $F \cap L_1$. 

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