EXTREME POINTS AND $l_1(\Gamma)$-SPACES

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Abstract. Let $X$ be a nontrivial real Banach space and let $E_X$ denote the set of extreme points of the closed unit ball $B(X)$.

Theorem 1. $X$ is an $l_1(\Gamma)$-space if and only if (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) $B(X)$ is the norm closed convex hull of $E_X$.

Theorem 2. Let $X = Y^*$. If (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) \{$e \in E_X: e(y) = 1$\} is countable for each $y$ in $Y$ with $\|y\| = 1$, then $X$ is an $l_1(\Gamma)$-space.

By definition, an $L$-projection on a Banach space $X$ is a projection $P$ such that $\|x\| = \|Px\| + \|x - Px\|$ for every $x$ in $X$; the range of $P$ is called an $L$-summand in $X$. An $l_1(\Gamma)$-space is a Banach space which is linearly isometric to the space $l_1(\Gamma)$ of all real-valued summable functions on some set $\Gamma$. Let $X$ be a nontrivial real Banach space and let $E_X$ denote the set of extreme points of the closed unit ball $B(X)$. In this paper we prove (Theorem 1) that $X$ is an $l_1(\Gamma)$-space if and only if (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) $B(X)$ is the norm closed convex hull of $E_X$. As a consequence we have (Theorem 2) that a dual space $X = Y^*$ is an $l_1(\Gamma)$-space if (i) span($e$) is an $L$-summand in $X$ for every $e \in E_X$ and (ii) \{$e \in E_X: e(y) = 1$\} is countable for each $y$ in $Y$ with $\|y\| = 1$. The proof of Theorem 2 uses the Bishop-Phelps theorem and a result of J. Bourgain to show that $B(X)$ is the norm closed convex hull of $E_X$. Our paper concludes with an example of a nonseparable space $Y$ which satisfies the hypotheses of Theorem 2 and contains uncountably many $y$ such that $\|y\| = 1$ and \{$e \in E_Y: e(y) = 1$\} is countably infinite.

In what follows, if $S$ is a subset of a Banach space, then the convex hull of $S$ is denoted by $co(S)$ and the linear span of $S$ by span $S$. The norm closure of $S$ is denoted by norm-cl($S$). All Banach spaces are assumed to be nontrivial.

In Lemmas 1 and 2, $A$ is a real Banach space for which $E_X \neq \emptyset$.

Lemma 1. Let $A$ be a nonempty finite subset of $E_X$ such that span($e$) is an $L$-summand in $X$ for every $e$ in $A$, and let $N = \text{span } A$. Then $B(N) = co(A \cup -A)$.

Proof. Since $N = \Sigma \text{span} (e)$ ($e \in A$), we have that $N$ is an $L$-summand in $X$ and $E_N = A \cup -A$ [1, Propositions 1.13 and 1.15]. Then $B(N) = co(A \cup -A)$ because $A$ is finite.

The following result was communicated to the author by Ulf Uttersrud.
Lemma 2. Assume that \( \text{span}(e) \) is an \( L \)-summand in \( X \) for every \( e \) in \( E_X \). Let \( \{e_n: n = 1, 2, \ldots \} \) be a linearly independent subset of \( E_X \) and let \( x_n \in \text{span}(e_n) \) for \( n = 1, 2, \ldots \). If \( \sum \|x_n\| < \infty \), then \( \sum x_n \) converges and \( \|\sum x_n\| = \sum \|x_n\| \).

Proof. The proof follows from the fact that \( \|\sum_{n=1}^k x_n\| = \sum_{n=1}^k \|x_n\| \) for all \( k \). To obtain the induction step, observe that if \( P \) is the \( L \)-projection of \( X \) onto \( N_k = \sum_{n=1}^k \text{span}(e_n) \), then \( Pe_{k+1} = 0 \) because \( e_{k+1} \notin N_k \) by Lemma 1. (An \( L \)-projection maps an extreme point to itself or 0.)

Theorem 1. A real Banach space \( X \) is an \( l_1(\Gamma) \)-space if and only if (i) \( \text{span}(e) \) is an \( L \)-summand in \( X \) for every \( e \) in \( E_X \) and (ii) \( B(X) \) is the norm closed convex hull of \( E_X \).

Proof. Suppose that \( X \) is an \( l_1(\Gamma) \)-space. We may assume that \( X = l_1(\Gamma) \), where \( \Gamma \) is a nonempty set. For each \( \gamma \) in \( \Gamma \) let \( \delta_\gamma \) be the characteristic function of \( \{\gamma\} \). Then \( E_X = \{\pm \delta_\gamma: \gamma \in \Gamma\} \). For each \( \gamma \) in \( \Gamma \), the map \( x \mapsto x\delta_\gamma \) is an \( L \)-projection of \( X \) onto \( \text{span}(\delta_\gamma) \). Thus condition (i) holds (as it does in any \( L_1 \)-space). To prove (ii), let \( x \in X \) with \( \|x\| < 1 \). Then there is a countable set \( \{\gamma_n\} \subseteq \Gamma \) such that \( x(\gamma_i) = 0 \) for \( \gamma \notin \{\gamma_n\} \) and \( \sum_{n=1}^\infty |x(\gamma_n)| < 1 \). For each \( k \) let \( x_k = \sum_{n=1}^k x(\gamma_n)\delta_{\gamma_n} \). Then \( \|x_k\| < 1 \) and hence by Lemma 1, \( x_k \in \text{co} E_X \). Therefore \( x \in \text{norm-cl(}co E_X) \).

For the converse, assume that (i) and (ii) are true. Let \( \Gamma \) be a maximal linearly independent subset of \( E_Y \). Then \( E_X = \Gamma \cup -\Gamma \). To see this, suppose there is \( e \in E_X \) with \( e \notin \Gamma \cup -\Gamma \). Then \( e \) is a linear combination of the elements of a finite subset \( A \) of \( \Gamma \). By Lemma 1, \( e \in \text{co}(A \cup -A) \). Then \( e \in A \cup -A \) since \( e \in E_X \), and we have a contradiction. If \( \Gamma = \{e_\gamma\} \), define an operator \( T: l_1(\Gamma) \rightarrow X \) by \( T(f) = \sum \gamma f(\gamma) e_\gamma \). By Lemma 2, \( T \) is an isometry. Hence its range is closed. By (ii) and the fact that \( E_X = \Gamma \cup -\Gamma \), the range of \( T \) is dense in \( X \). Thus \( T \) is surjective.

Theorem 2. Let \( Y \) be a real Banach space such that (i) \( \text{span}(e) \) is an \( L \)-summand in \( Y^* \) for every \( e \) in \( E_{Y^*} \), and (ii) \( \{e \in E_{Y^*}: e(y) = 1\} \) is countable for each \( y \) in \( Y \) with \( \|y\| = 1 \).

Then \( Y^* \) is an \( l_1(\Gamma) \)-space.

Proof. By Theorem 1 it suffices to show that \( B(Y^*) \) is the norm closed convex hull of \( E_{Y^*} \). Let \( f \in B(Y^*) \) with \( f \neq 0 \). By the Bishop-Phelps theorem [2], the set of those \( g \) in \( Y^* \) which attain their norm is dense in \( Y^* \). Hence given \( \epsilon > 0 \), there is \( g \) in \( Y^* \) such that \( \|f - g\| < \epsilon \|g\| \) for all \( g \in B(Y^*) \) with \( \|y\| = 1 \). Let \( F_y = \{h \in B(Y^*): h(y) = 1\} \). Then \( F_y \) is a weak* compact convex set and \( \|g\| < \epsilon \). Let \( E_y \) denote the set of extreme points of \( F_y \). Then \( E_y \subseteq E_{Y^*} \) because \( F_y \) is an extremal subset of \( B(Y^*) \). Thus \( E_y = \{e \in E_{Y^*}: e(y) = 1\} \). Then \( F_y = \text{norm-cl(}co E_y) \) because \( E_y \) is countable [3]. Let \( h \in \text{co} E_y \) with \( \|h - g\| < \epsilon \). Then \( \|h - f\| < 2\epsilon \), hence \( \|f\| < 2\epsilon \).

Since \( \|f\| < \epsilon \), it follows that \( f \in \text{norm-cl(}co E_{Y^*}) \).

We now give an example of a space \( Y \) which satisfies the hypotheses of Theorem 2 and contains uncountably many \( y \) such that \( \|y\| = 1 \) and \( \{e \in E_{Y^*}: e(y) = 1\} \) is countably infinite.
Let $T$ denote the set of all ordinals less than or equal to the first uncountable ordinal $\Omega$, and let $T$ have the order topology. Let $Y = \{ f \in C(T) : f(\Omega) = 0 \}$. Then $Y^*$ is an $L$-space because $Y$ is an $M$-space; hence the first hypothesis of Theorem 2 is satisfied. For each $t$ in $T$, let the evaluation functional $e_t$ be defined on $Y$ by $e_t (f) = f(t)$ for all $f$ in $Y$. Then $E_{Y^*} = \{ \pm e_t : t \in T, t \neq \Omega \}$. Since each function in $C(T)$ is eventually constant, the second hypothesis of Theorem 2 is satisfied. For each $t$ in $T$ such that $\omega \leq t < \Omega$, let $f_t$ be the characteristic function of the interval $[0, t]$. Then $f_t \in Y$, $\| f_t \| = 1$, and $\{ e \in E_{Y^*} : e(f_t) = 1 \}$ is countably infinite. Clearly the set of functions $f_t$ is uncountable.

In conclusion, we remark that $C(T)^* = l_1(T)$ [4, p. 175], hence the converse of Theorem 2 is false.

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