EXTREME POINTS AND $l_1(\Gamma)$-SPACES

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Abstract. Let $X$ be a nontrivial real Banach space and let $E_X$ denote the set of extreme points of the closed unit ball $B(X)$.

**Theorem 1.** $X$ is an $l_1(\Gamma)$-space if and only if (i) span$(e)$ is an $L$-summand in $X$ for every $e$ in $E_X$ and (ii) $B(X)$ is the norm closed convex hull of $E_X$.

**Theorem 2.** Let $X = Y^*$. If (i) span$(e)$ is an $L$-summand in $X$ for every $e$ in $E_X$ and (ii) $\{e \in E_X: e(y) = 1\}$ is countable for each $y$ in $Y$ with $\|y\| = 1$, then $X$ is an $l_1(\Gamma)$-space.

By definition, an $L$-projection on a Banach space $X$ is a projection $P$ such that $\|x\| = \|Px\| + \|x - Px\|$ for every $x$ in $X$; the range of $P$ is called an $L$-summand in $X$. An $l_1(\Gamma)$-space is a Banach space which is linearly isometric to the space $l_1(\Gamma)$ of all real-valued summable functions on some set $\Gamma$. Let $X$ be a nontrivial real Banach space and let $E_X$ denote the set of extreme points of the closed unit ball $B(X)$. In this paper we prove (Theorem 1) that $X$ is an $l_1(\Gamma)$-space if and only if (i) span$(e)$ is an $L$-summand in $X$ for every $e$ in $E_X$ and (ii) $B(X)$ is the norm closed convex hull of $E_X$. As a consequence we have (Theorem 2) that a dual space $X = Y^*$ is an $l_1(\Gamma)$-space if (i) span$(e)$ is an $L$-summand in $X$ for every $e$ in $E_X$ and (ii) $\{e \in E_X: e(y) = 1\}$ is countable for each $y$ in $Y$ with $\|y\| = 1$. The proof of Theorem 2 uses the Bishop-Phelps theorem and a result of J. Bourgain to show that $B(X)$ is the norm closed convex hull of $E_X$. Our paper concludes with an example of a nonseparable space $Y$ which satisfies the hypotheses of Theorem 2 and contains uncountably many $y$ such that $\|y\| = 1$ and $\{e \in E_Y: e(y) = 1\}$ is countably infinite.

In what follows, if $S$ is a subset of a Banach space, then the convex hull of $S$ is denoted by co$S$ and the linear span of $S$ by span$S$. The norm closure of $S$ is denoted by norm-cl$(S)$. All Banach spaces are assumed to be nontrivial.

In Lemmas 1 and 2, $X$ is a real Banach space for which $E_X \neq \emptyset$.

**Lemma 1.** Let $A$ be a nonempty finite subset of $E_X$ such that span$(e)$ is an $L$-summand in $X$ for every $e$ in $A$, and let $N = \text{span} A$. Then $B(N) = \text{co}(A \cup -A)$.

**Proof.** Since $N = \sum \text{span}(e)$ ($e \in A$), we have that $N$ is an $L$-summand in $X$ and $E_N = A \cup -A$ [1, Propositions 1.13 and 1.15]. Then $B(N) = \text{co}(A \cup -A)$ because $A$ is finite.

The following result was communicated to the author by Ulf Uttersrud.
Lemma 2. Assume that span(e) is an L-summand in X for every e in E_x. Let \{e_n: n = 1, 2, \ldots\} be a linearly independent subset of E_x and let x_n \in \text{span}(e_n) for n = 1, 2, \ldots. If \|\Sigma x_n\| < \infty, then \Sigma x_n converges and \|\Sigma x_n\| = \|\Sigma x_n\|.

Proof. The proof follows from the fact that \|\Sigma_{k=1}^n x_n\| = \Sigma_{n=1}^k \|x_n\| for all k. To obtain the induction step, observe that if P is the L-projection of X onto N_k = \Sigma_{n=1}^k \text{span}(e_n), then佩_{k+1} = 0 because e_{k+1} \notin N_k by Lemma 1. (An L-projection maps an extreme point to itself or 0.)

Theorem 1. A real Banach space X is an l_1(\Gamma)-space if and only if (i) span(e) is an L-summand in X for every e in E_x and (ii) B(X) is the norm closed convex hull of E_x.

Proof. Suppose that X is an l_1(\Gamma)-space. We may assume that X = l_1(\Gamma), where \Gamma is a nonempty set. For each \gamma in \Gamma let \delta_\gamma be the characteristic function of \{\gamma\}. Then E_x = {\pm \delta_\gamma: \gamma \in \Gamma}. For each \gamma in \Gamma, the map x \mapsto x\delta_\gamma is an L-projection of X onto span(\delta_\gamma). Thus condition (i) holds (as it does in any L_1-space). To prove (ii), let x \in X with \|x\| \leq 1. Then there is a countable set \{\gamma_n\} \subseteq \Gamma such that x(\gamma) = 0 for \gamma \notin \{\gamma_n\} and \Sigma_{n=1}^\infty |x(\gamma_n)| < 1. Then x = \Sigma_{n=1}^\infty x(\gamma_n)\delta_{\gamma_n}. For each k let x_k = \Sigma_{n=1}^k x(\gamma_n)\delta_{\gamma_n}. Then \|x_k\| \leq 1 and hence by Lemma 1, x_k \in \text{co}(E_x). Therefore x \in \text{norm-cl}(\text{co}(E_x)).

For the converse, assume that (i) and (ii) are true. Let \Gamma be a maximal linearly independent subset of E_y. Then E_x = \Gamma \cup -\Gamma. To see this, suppose there is e \in E_x with e \notin \Gamma \cup -\Gamma. Then e is a linear combination of the elements of a finite subset A of \Gamma. By Lemma 1, e \in \text{co}(A \cup -A). Then e \notin A \cup -A since e \in E_x, and we have a contradiction. If \Gamma = \{e_\gamma\}, define an operator T: l_1(\Gamma) \to X by T(f) = \Sigma f(\gamma) e_\gamma. By Lemma 2, T is an isometry. Hence its range is closed. By (ii) and the fact that E_x = \Gamma \cup -\Gamma, the range of T is dense in X. Thus T is surjective.

Theorem 2. Let Y be a real Banach space such that (i) span(e) is an L-summand in Y* for every e in E_y, and (ii) \{e \in E_y: e(y) = 1\} is countable for each y in Y with \|y\| = 1. Then Y* is an l_1(\Gamma)-space.

Proof. By Theorem 1 it suffices to show that B(Y*) is the norm closed convex hull of E_{Y*}. Let f \in B(Y*) with f \neq 0. By the Bishop-Phelps theorem [2], the set of those g in Y* which attain their norm is dense in Y*. Hence given \epsilon > 0, there is g in Y* such that \|f\| \|g - f\| < \epsilon and \|g\| = g(y), where y \in Y with \|y\| = 1. Let F_y = \{h \in B(Y*): h(y) = 1\}. Then F_y is a weak* compact convex set and g/\|g\| \in F_y. Let E_y denote the set of extreme points of F_y. Then E_y \subseteq E_{Y*} because F_y is an extremal subset of B(Y*). Thus E_y = {e \in E_{Y*}: e(y) = 1}. Then F_y = \text{norm-cl}(\text{co}(E_y)) because E_y is countable [3]. Let h \in \text{co}(E_y) with \|h - g\| < \epsilon. Then \|h - f\| < 2\epsilon, hence \|f\| \|h - f\| < 2\epsilon \|f\|.

Since \|f\| \in \text{co}(E_y \cup -E_y), it follows that f \in \text{norm-cl}(\text{co}(E_{Y*})). We now give an example of a space Y which satisfies the hypotheses of Theorem 2 and contains uncountably many y such that \|y\| = 1 and \{e \in E_{Y*}: e(y) = 1\} is countably infinite.
Let \( T \) denote the set of all ordinals less than or equal to the first uncountable ordinal \( \Omega \), and let \( T \) have the order topology. Let \( Y = \{ f \in C(T): f(\Omega) = 0 \} \). Then \( Y^* \) is an \( L \)-space because \( Y \) is an \( M \)-space; hence the first hypothesis of Theorem 2 is satisfied. For each \( t \) in \( T \), let the evaluation functional \( e_t \) be defined on \( Y \) by \( e_t(f) = f(t) \) for all \( f \) in \( Y \). Then \( E_{Y^*} = \{ \pm e_t: t \in T, t \neq \Omega \} \). Since each function in \( C(T) \) is eventually constant, the second hypothesis of Theorem 2 is satisfied. For each \( t \) in \( T \) such that \( \omega < t < \Omega \), let \( f_t \) be the characteristic function of the interval \([0, t]\). Then \( f_t \in Y \), \( \| f_t \| = 1 \), and \( \{ e \in E_{Y^*}: e(f_t) = 1 \} \) is countably infinite. Clearly the set of functions \( f_t \) is uncountable.

In conclusion, we remark that \( C(T)^* = l_1(T) \) [4, p. 175], hence the converse of Theorem 2 is false.

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REFERENCES