ELEMENTARY PROOFS OF SOME ASYMPTOTIC RADIAL UNIQUENESS THEOREMS

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Abstract. Elementary proofs of several generalizations of Tse's extension of an asymptotic radial uniqueness theorem of Barth and Schneider are given.

Let $\Delta = \{|z|<1\}$ and $C = \{|z|=1\}$. The following is an extension to meromorphic functions by Tse [5] of a theorem of Barth and Schneider [1].

**Theorem 1.** Let $\mu$ be a positive monotone decreasing function with domain $[0,1)$ such that $\lim_{r \to 1^-} \mu(r) = 0$. Let $S$ be a second category subset of $C$. If $f$ is a meromorphic function on $\Delta$ with the property that $f(\eta r) = o[\mu(r)]$ for each $\eta \in S$, then $f = 0$.

Barth and Schneider's proof (for bounded analytic functions) depends on deep theorems of Mergelyan, Lusin-Privalov, and Collingwood. Tse's proof of Theorem 1 is based on the method of Barth and Schneider. An equivalent formulation [4, Corollary 2] of Theorem 1 may be obtained in a more straightforward manner as a corollary of a theorem of Rippon [4, Theorem 1]; however, this proof still relies on the Collingwood maximality theorem as well as results and methods needed for a proof of the Lusin-Privalov theorem. The proofs that we shall give to several generalizations of Theorem 1 are elementary and are based on the following.

**Category Principle.** Let $S$ be a second category set. If $S = \bigcup_{n=1}^{\infty} F_n$ with each $F_n$ closed, then some $F_n$ contains a nonempty open set.

**Proof.** Since $E$ is of second category, some $F_n$ must be dense in an open set $U$. Since $F_n$ is closed, we have $U \subseteq F_n$ as required.

We turn now to our first generalization of Theorem 1. Let $\Theta$ be a continuum contained in $\Delta = \{|z|\leq 1\}$ such that $\Theta \cap C = \{1\}$ and let $\Theta_\eta = \{z: z \in \Theta\}$ for each $\eta \in C$. For $\mu$ a positive function with domain $[0,1)$ such that $\lim_{r \to 1^-} \mu(r) = 0$ and $\eta \in C$, we shall write $f(z) = O[\mu(|z|)]$, $z \in \Theta_\eta$, when $\limsup_{|z| \to 1^-} |f(z)|/|\mu(|z|)| < +\infty$, $z \in \Theta_\eta \cap \Delta$.

**Theorem 2.** Let $\mu$ be a positive function with domain $[0,1)$ such that $\lim_{r \to 1^-} \mu(r) = 0$ and $S$ a second category subset of $C$. If $f$ is a meromorphic function on $\Delta$ with the property that $f(z) = O[\mu(|z|)]$, $z \in \Theta_\eta$, for each $\eta \in S$, then $f \equiv 0$. 

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Theorem 1 follows from Theorem 2 when \( \emptyset = [0, 1] \).

**Proof.** Let \( Z_n(f) = \{ \eta \in C : f(z) = O[\mu(|z|)], z \in \emptyset_n \} \). Then \( Z_n(f) = \bigcup_{n=1}^{\infty} F_n \) where \( F_n = \{ \eta \in C : |f(z)| < \eta \mu(|z|), z \in \emptyset_n \text{ and } 1 - \frac{1}{n} \leq |z| < 1 \} \) for each \( n \). Now each \( F_n \) is closed by the continuity of \( f \). Since \( Z_n(f) \) is of second category by assumption, there exists some \( n \) for which \( F_n \) contains a nonempty open arc \( A \) (category principle). Note that the set \( \bigcup_{\eta \in A} \emptyset_n \cap \{ 1 - \frac{1}{n} \leq |z| < 1 \} \) is a neighborhood (in \( A \)) of each point of \( A \). It follows from the definition of \( F_n \) that \( f \) is continuously 0 at each point of the arc \( A \). We conclude from the Schwarz reflection principle and the identity theorem that \( f \equiv 0 \). Theorem 2 is established.

Our next generalization extends Theorem 1 to the unit ball in \( C^n, n \geq 1 \). Let \( A_n = \{ z \in C^n : \|z\| < 1 \} \) and \( C_n = \{ z \in C^n : \|z\| = 1 \} \).

**Theorem 3.** Let \( \mu \) be as in Theorem 2 and \( S \) a second category subset of \( C_n \). If \( f \) is meromorphic on \( A_n \) with \( f(\tau \eta) = O[\mu(\tau)] \) for each \( \eta \in S \), then \( f \equiv 0 \).

**Proof.** Except for the modification that \( A \) is now an open subset of \( C_n \) instead of an open arc of \( C \), the proof proceeds as above (with \( \emptyset = [0, 1] \)) up to the conclusion that \( f \) is continuously 0 at each point of \( A \).

We show that \( f \equiv 0 \) as follows. Let \( w \in A_n \) and \( \eta \in A \). There exists a univalent analytic map \( \varphi \) defined on a neighborhood of \( \Delta_n \) mapping \( \Delta_n \) onto itself such that \( \varphi(w) = (0, \ldots, 0) \) and \( \varphi(\eta) = (1, 0, \ldots, 0) \). (Such a map \( \varphi \) can be constructed explicitly using maps of the form

\[
\psi_{\beta} : (z_1, \ldots, z_n) \rightarrow \left( \frac{z_1 - \beta}{1 - \beta z_1}, \frac{\sqrt{1 - |\beta|^2}}{1 - \beta z_1} z_2, \ldots, \frac{\sqrt{1 - |\beta|^2}}{1 - \beta z_1} z_n \right),
\]

\((z_1, \ldots, z_n) \in \Delta_n \) for \( \beta \in \Delta \) and unitary linear transformations; cf. [2, p. 420].) Then \( g(z) = f \circ \varphi^{-1}(z, 0, \ldots, 0) \) is a meromorphic function on \( \Delta \) which is continuously 0 at each point of an open arc \( I \) containing 1 and contained in \( \{ \xi \in C : (\xi, 0, \ldots, 0) \in \varphi(A) \} \). It follows from the Schwarz reflection principle and the identity theorem that \( g \equiv 0 \). In particular, \( f(w) = f \circ \varphi^{-1}(0, \ldots, 0) = g(0) = 0 \). Since \( w \in A_n \) was arbitrary, we conclude that \( f \equiv 0 \). This completes the proof of Theorem 3.

It is possible to generalize Theorem 3 in a way analogous to that in which Theorem 2 generalizes Theorem 1, though care must be taken in framing a workable definition of rotating a continuum when \( n > 1 \). One possibility is to phrase such a definition in terms of group actions on the sphere \( C_n \). An analogue of such a generalization for a half space \( H_n = \{ (z_1, \ldots, z_n) \in C^n : \text{Im } z_n > 0 \} \) is more easily formulated and proved; however, we shall not pursue these generalizations here.

Rippon [3, Theorem 3] has given a subharmonic analogue of Theorem 1 for the half space \( D = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \} \) when \( n > 2 \). His proof depends on a generalized form of the Collingwood maximality theorem for “fine continuous” functions (a class containing the subharmonic functions) proved in the same paper [3, Theorem 1]. It is also noted that his arguments apply equally well to the half plane. The following analogue of Rippon’s result for continuous subharmonic functions in the disk \( \Delta \) may be proved along the same lines as our proof of Theorem 2. We assume that \( \emptyset \) and \( \emptyset_n, \eta \in C \), are as preceding that theorem.
Theorem 4. Let $v$ be a real-valued function with domain $[0, 1)$ such that $\lim_{r \to 1} v(r) = -\infty$. Let $u$ be a continuous subharmonic function on $\Delta$. If $S$ is a second category subset of $C$ such that

\[ \limsup_{z \to 1} u(z) - v(|z|) < +\infty, \quad \eta \in S, \]

\[ z \in \Theta_{\eta} \cap \Delta \]

then $u \equiv -\infty$.

For the proof, note that if a subharmonic function $u$ on $\Delta$ is continuously $-\infty$ at each point of a nonempty open arc $A$ of $C$, then $u \equiv -\infty$ as is seen using simple harmonic measure estimates. When $f$ is analytic, Theorem 2 is easily subsumed under Theorem 4. In fact, letting $\nu = \log \mu$, $\nu = \log |f|$, and observing that $f(z) = O(\mu(|z|))$, $z \in \Theta_{\eta}$, for each $\eta \in S$ implies (1), we see that the conclusion $u \equiv -\infty$ guarantees that $f \equiv 0$. Finally, we remark that in the case when $\Theta$ is the image of a Jordan arc $\gamma$ such that $|\gamma|$ is strictly increasing, Theorems 2 and 4 are seen to be sharp when $\mu$ and $\nu$ are monotonic using only a slight modification of the construction used by P. Gauthier (see [5, Theorem B]) to show that Theorem 1 is sharp.

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