DISTANCE ESTIMATES FOR VON NEUMANN ALGEBRAS

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Abstract. It is shown that for certain von Neumann algebras & there is a constant C such that

$$\text{dist}(T, &) \leq C \sup_{P \in \text{lat } &} \| P^\perp TP \|$$

for all T in \( \mathcal{B}(\mathcal{H}) \).

1. Introduction. Throughout this paper, \( \mathcal{H} \) denotes a separable Hilbert space and \( \mathcal{B}(\mathcal{H}) \) is the algebra of all bounded linear operators on \( \mathcal{H} \). For any subalgebra \( \mathfrak{A} \) of \( \mathcal{B}(\mathcal{H}) \), let \( \text{lat } \mathfrak{A} \) denote the lattice of orthogonal projections \( P \) invariant for \( \mathfrak{A} \). That is, \( P^\perp AP = 0 \) for all \( A \) in \( \mathfrak{A} \), where \( P = I - P \). \( \mathfrak{A} \) is said to be reflexive if every operator \( B \) satisfying \( P^\perp BP = 0 \) for all \( P \) in \( \text{lat } \mathfrak{A} \) belongs to \( \mathfrak{A} \).

Let \( \mathfrak{A} \) be a reflexive algebra and \( T \) an arbitrary operator in \( \mathcal{B}(\mathcal{H}) \). It is easy to see that

$$\text{dist}(T, \mathfrak{A}) \geq \sup_{P \in \text{lat } \mathfrak{A}} \| P^\perp TP \|.$$ 

Arveson [1] proved that if \( \mathfrak{A} \) is a nest algebra, then equality actually occurs in (1). Davidson [3] has referred to Choi's example which shows that equality fails to hold even if \( \mathcal{H} \) is finite dimensional and \( \mathfrak{A} \) is a m.a.s.a. He asked: if \( \mathfrak{A} \) is reflexive and \( \text{lat } \mathfrak{A} \) is commutative, then is there a constant \( C \) such that

$$\text{dist}(T, \mathfrak{A}) \leq C \sup_{P \in \text{lat } \mathfrak{A}} \| P^\perp TP \|$$

for all \( T \) in \( \mathcal{B}(\mathcal{H}) \)?

In this paper, we shall prove that (2) holds (with \( C = 2 \)) if \( \mathfrak{A} \) is a von Neumann algebra such that either \( \mathfrak{A} \) or \( \mathfrak{A}' \) is abelian. Note that if \( \mathfrak{A}' \) is commutative, then so is \( \text{lat } \mathfrak{A} \). Also if \( \mathfrak{A} \) is a weakly closed unital algebra of normal operators, then (2) holds with \( C = 3 \).

In [5], Johnson conjectured that for a von Neumann algebra \( \mathfrak{A} \) there is a positive constant \( K \) such that for all \( T \) in \( \mathcal{B}(\mathcal{H}) \),

$$\text{dist}(T, \mathfrak{A}) \leq K \| \Delta_T \|_{\mathfrak{A}}$$

where \( \Delta_T \) is the derivation \( \Delta_T(S) = ST - TS \). Christensen [2] proved that (3) holds for a very large class of von Neumann algebras. We will show that (2) and (3) are equivalent.

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2. Distance estimates and derivations.

THEOREM 2.1. Let $\mathcal{B}$ be a von Neumann algebra and let $T$ belong to $\mathcal{B}(\mathcal{K})$.

(i) If $\text{dist}(T, \mathcal{B}) \leq K \|\Delta_T\|_{\mathcal{B}}$, then
$$\text{dist}(T, \mathcal{B}) \leq 4K \sup_{P \in \text{lat } \mathcal{B}} \|P^\perp TP\|.$$  

(ii) If $\text{dist}(T, \mathcal{B}) \leq C \sup_{P \in \text{lat } \mathcal{B}} \|P^\perp TP\|$, then
$$\text{dist}(T, \mathcal{B}) \leq \frac{C}{2} \|\Delta_T\|_{\mathcal{B}}.$$  

PROOF. Since $\text{lat } \mathcal{B}$ is complemented, we have
$$2 \sup_{P \in \text{lat } \mathcal{B}} \|P^\perp TP\| = 2 \sup_{P \in \text{lat } \mathcal{B}} \max \{\|P^\perp TP\|, \|PTP^\perp\|\}$$
$$= 2 \sup_{P \in \text{lat } \mathcal{B}} \|PT - TP\|$$
$$= \sup_{P \in \text{lat } \mathcal{B}} \| (2P - I)T - T(2P - I) \| \leq \|\Delta_T\|_{\mathcal{B}}.$$  

This proves (ii).

Now suppose $\sup_{P \in \text{lat } \mathcal{B}} \|PT - TP\| = \delta$. Then for $P$ in lat $\mathcal{B}$, $\|\Delta_T(2P - I)\| \leq 2\delta$. Let $\mathcal{H}$ be the real vector space of all Hermitian operators in $\mathcal{B}'$. By the Krein-Milman theorem, the unit ball of $\mathcal{H}$ is the weakly closed convex hull of its extreme points. But these extreme points are precisely $(2P - I)$ for projections $P$ in lat $\mathcal{B}$, namely lat $\mathcal{B}$. Thus $\|\Delta_T\|_{\mathcal{H}} \leq 2\delta$. If $B$ in $\mathcal{B}'$ has $\|B\| \leq 1$, then write $B = A_1 + iA_2$, where $A_i$ are Hermitian and $\|A_i\| \leq 1$. Then $\|\Delta_T(B)\| \leq \|\Delta_T(A_1)\| + \|\Delta_T(A_2)\| \leq 4\delta$, which proves (i).  

3. Abelian von Neumann algebras. In [2], it is established that if $\mathcal{B}'$ is abelian,
$$\text{dist}(T, \mathcal{B}) \leq \|\Delta_T\|_{\mathcal{B}}.$$  

By Theorem 2.1, we conclude that (1) holds with $C = 4$. In fact, we have

LEMMA 3.1. If $\mathcal{B}$ is a von Neumann algebra with abelian commutant, then for every $T$ in $\mathcal{B}(\mathcal{K})$,
$$\text{dist}(T, \mathcal{B}) \leq 2 \sup_{P \in \text{lat } \mathcal{B}} \|P^\perp TP\|.$$  

PROOF. Let $\delta = \sup_{P \in \text{lat } \mathcal{B}} \|P^\perp TP\|$. Let $G$ be the group of unitaries generated by $\{2P - I: P \in \text{lat } \mathcal{B}\}$. Then for every element $U$ in $G$, $\|UT - TU\| \leq 2\delta$, whence $\|T - U^{-1}TU\| \leq 2\delta$. Since $G$ is abelian, it has an invariant mean $m$. Let $f(U) = U^{-1}TU$, and define $T_0 = m(f)$ following the method of [6]. Since $T_0$ is in the weakly closed convex hull of $\{U^{-1}TU: U \in G\}$, we have $\|T - T_0\| \leq 2\delta$. But the invariance of $m$ shows that $T_0$ belongs to $G' = (\mathcal{B}')' = \mathcal{B}$. Thus dist($T, \mathcal{B}$) $\leq 2\delta$.  

LEMMA 3.2. Let $\phi$ be a functional on $\mathcal{B}(\mathcal{K})$ continuous in the weak operator topology such that $\phi(I) = 0$. Then the kernel of $\phi$ contains a m.a.s.a.

PROOF. There is a finite rank operator $A$ such that $\phi(T) = \text{tr}(AT)$ for all $T$ in $\mathcal{B}(\mathcal{K})$. It suffices to find an orthonormal basis $\{f_i\}$ for $\mathcal{K}$ such that $(Af_i, f_i) = 0$ for all $i$. For then, we take our m.a.s.a. to be all operators which are diagonal with respect to this basis.
The numerical range of an operator $B$, namely $W(B) = \{(Bx, x) : \|x\| = 1\}$, is always convex and nonempty. When $B$ acts on a space of finite dimension $n$, we have

$$\frac{1}{n} \sum_{i=1}^{n} (Be_i, e_i) = \frac{1}{n} \text{tr} B.$$  

So if $\text{tr} B = 0$, then 0 belongs to $W(B)$ and thus there is a unit vector $f$ such that $(Bf, f) = 0$. On the complement $\mathcal{M}$ of span$(f)$, we have

$$0 = \text{tr} B = (Bf, f) + \text{tr}(B |_{\mathcal{M}^\perp}) = \text{tr}(B |_{\mathcal{M}^\perp}).$$

By induction, there is an orthonormal basis $\{f_i, 1 \leq i \leq n\}$ with $(Bf_i, f_i) = 0$.

Choose a finite dimensional subspace $\mathcal{H}$ which reduces $A$ and $A |_{\mathcal{M}^\perp} = 0$. Apply the previous paragraph to $A |_{\mathcal{M}}$ and complete the orthonormal set with an arbitrary basis for $\mathcal{M}^\perp$. □

**Lemma 3.3.** Let $\mathcal{A}$ be an abelian von Neumann algebra on $\mathcal{H}$, and let $\phi$ be a weak operator continuous functional on $\mathcal{B}(\mathcal{H})$ which annihilates $\mathcal{A}$. Then there is a m.a.s.a. $\mathcal{M}$ containing $\mathcal{A}$ in the kernel of $\phi$.

**Proof.** We shall use the direct integral decomposition of $\mathcal{A}$ [8, p. 19]. There is a measure space $(Z, \mu)$ such that $\mathcal{H} = \int \mathcal{H}(\xi) d\mu(\xi)$ and $\mathcal{A}$ is the algebra of all operators $T = \int T(\xi) d\mu(\xi)$ where $T(\xi)$ is a scalar multiple of the identity $I_{\mathcal{H}(\xi)}$. We can write $\phi$ in the form $\phi(T) = \sum_{i=1}^{n} (T(x_i, y_i))$, and $x_i = \int x_i(\xi) d\mu(\xi)$ and $y_i = \int y_i(\xi) d\mu(\xi)$.

Define $f(\xi) = \sum_{i=1}^{n} (x_i(\xi), y_i(\xi))$. Then $f(\xi)$ belongs to $L^1(Z, \mu)$. If $g(\xi)$ is a bounded measurable function on $(Z, \mu)$, then

$$0 = \phi\left(\int \phi g(\xi) I_{\mathcal{H}(\xi)} d\mu(\xi)\right) = \int f(\xi) g(\xi) d\mu(\xi).$$

Hence $f(\xi) = 0$ a.e.

Let $\phi_f$ be the weak operator continuous functional on $\mathcal{B}(\mathcal{H}(\xi))$ given by $\phi_f(S) = \sum_{i=1}^{n} (x_i(\xi), y_i(\xi))$. Then $\phi_f(t) = 0$ for almost all $\xi$. So by Lemma 3.2, there is a m.a.s.a. $\mathcal{M}(\xi)$ in the kernel of $\phi_f$. Let $\mathcal{M} = \int \mathcal{M}(\xi) d\mu(\xi)$. It is easy to verify that $\mathcal{M}$ is a m.a.s.a. in $\mathcal{B}(\mathcal{H})$ and $\phi(\mathcal{M}) = 0$. □

**Lemma 3.4.** If $\mathcal{A}$ is an abelian von Neumann algebra, and $T$ belongs to $\mathcal{B}(\mathcal{H})$, then

$$(4) \quad \text{dist}(T, \mathcal{A}) = \sup \text{dist}(T, \mathcal{M})$$

where the sup is taken over all m.a.s.a.'s $\mathcal{M}$ containing $\mathcal{A}$.

**Proof.** We will prove the nontrivial inequality $\text{dist}(T, \mathcal{A}) \geq \sup \text{dist}(T, \mathcal{M})$. Let $\delta$ denote the right-hand side of (4), and fix $\epsilon > 0$. For each m.a.s.a. $\mathcal{M}$ containing $\mathcal{A}$, choose $T_{\mathcal{M}}$ in $\mathcal{M}$ with $\|T - T_{\mathcal{M}}\| < \delta + \epsilon$. Let $\mathcal{K}$ denote the weak operator closed convex hull of $\{T_{\mathcal{M}}\}$. If $\mathcal{K}$ were disjoint from $\mathcal{A}$, then by the Hahn-Banach theorem, there is a weak operator continuous linear functional $\phi$ on $\mathcal{B}(\mathcal{H})$ which annihilates $\mathcal{A}$ but is nonzero on all of $\mathcal{K}$. But by Lemma 3.3, there is a m.a.s.a. $\mathcal{M}_0$ containing $\mathcal{A}$ in the kernel of $\phi$. In particular, $\phi(T_{\mathcal{M}_0}) = 0$ which is a contradiction. Hence $\mathcal{K}$ meets $\mathcal{A}$. Let $A$ belong to the intersection $\mathcal{K} \cap \mathcal{A}$. Then $\|T - A\| \leq \delta + \epsilon$. Since $\epsilon$ was arbitrary, $\text{dist}(T, \mathcal{A}) < \delta$. □
Theorem 3.5. If $A$ is a von Neumann algebra such that $\mathcal{E}$ or $\mathcal{E}'$ is abelian, then for all $T$ in $\mathfrak{B}(\mathbb{H})$,

$$\text{dist}(T, \mathcal{E}) \leq 2 \sup_{P \in \text{lat } \mathcal{E}} \|P^\perp TP\|.$$  

Proof. Lemma 3.1 suffices if $\mathcal{E}'$ is abelian. If $\mathcal{E}$ is abelian and $\mathfrak{R}$ is a m.a.s.a. containing $\mathcal{E}$, then $\text{lat } \mathfrak{R} \subset \text{lat } \mathcal{E}$, so by Lemma 3.4,

$$\text{dist}(T, \mathcal{E}) = \sup_{\mathfrak{R}} \text{dist}(T, \mathfrak{R}) \leq \sup_{\mathfrak{R}} 2 \sup_{P \in \text{lat } \mathfrak{R}} \|P^\perp TP\| \leq 2 \sup_{P \in \text{lat } \mathcal{E}} \|P^\perp TP\|. \quad \square$$

We conclude this section with a distance estimate for (possibly non-self-adjoint) algebras of normal operators.

Theorem 3.6. If $\mathfrak{B}$ is a unital weakly closed algebra of normal operators, then for all $T$ in $\mathfrak{B}(\mathbb{H})$,

$$\text{dist}(T, \mathfrak{B}) \leq 3 \sup_{P \in \text{lat } \mathfrak{B}} \|P^\perp TP\|.$$  

Proof. By [7, Lemma 9.20], $\mathfrak{B}$ is abelian. Let $\mathfrak{R}$ be a m.a.s.a. containing $\mathfrak{B}$. We may assume that $\|T\| = 1$. Let $T_0$ be the operator produced in the proof of Lemma 3.1. If $U$ is a unitary in $\mathfrak{R}$, and $P$ belongs to $\text{lat } \mathfrak{B}$, let $Q = UPU^*$. For $B$ in $\mathfrak{B}$,

$$Q^\perp BQ = U P^\perp (U^*BU)PU^* = U(P^\perp BP)U^* = 0.$$  

Thus $Q$ belongs to $\text{lat } \mathfrak{B}$. Now

$$\|P^\perp (U^*TU)P\| = \|(UP^\perp U^*)T(UPU^*)\| = \|Q^\perp TQ\|.$$  

Since $T_0$ belongs to the weakly closed convex hull of $\{U^*TU\}$,

$$\text{sup}_{P \in \text{lat } \mathfrak{B}} \|P^\perp T_0P\| \leq \text{sup}_{P \in \text{lat } \mathfrak{B}} \|P^\perp TP\|. \quad (5)$$

Also, by Lemma 3.1,

$$\|T - T_0\| \leq 2 \sup_{P \in \text{lat } \mathfrak{R}} \|P^\perp TP\| \leq 2 \sup_{P \in \text{lat } \mathfrak{B}} \|P^\perp TP\|.$$  

We can complete the proof by proving that $\text{dist}(T_0, \mathfrak{B}) \leq \sup_{P \in \text{lat } \mathfrak{B}} \|P^\perp T_0P\|$. Now $\text{dist}(T_0, \mathfrak{B}) = \sup |\phi(T_0)|$ where $\phi$ runs over all weak * continuous functionals on $\mathfrak{R}$ of norm one which vanish on $\mathfrak{B}$. Let $\varepsilon > 0$, and choose such a functional $\phi$ with $\text{dist}(T_0, \mathfrak{B}) < |\phi(T_0)| + \varepsilon$. Since $\mathfrak{R}$ is maximal abelian, there are vectors $x$ and $y$ such that $\phi(M) = (Mx, y)$ for all $M$ in $\mathfrak{R}$ and $\|x\| \|y\| \leq 1 + \varepsilon$. Let $P_0$ be the orthogonal projection onto the closed span of $\mathfrak{B}x$. Clearly, $P_0$ belongs to $\text{lat } \mathfrak{R}$, $P_0x = x$, and $P_0^\perp y = y$. So

$$\text{dist}(T_0, \mathfrak{B}) \leq |(T_0x, y)| + \varepsilon = |(P_0^\perp T_0P_0x, y)| + \varepsilon \leq \|P_0^\perp T_0P_0\| \|x\| \|y\| + \varepsilon \leq (1 + \varepsilon) \text{sup}_{P \in \text{lat } \mathfrak{B}} \|P^\perp T_0P\| + \varepsilon.$$  

Thus, $\text{dist}(T_0, \mathfrak{B}) \leq \sup_{P \in \text{lat } \mathfrak{B}} \|P^\perp T_0P\|$, and the theorem is proven. \quad \square

4. Extensions of derivations. Let $\mathcal{E}$ be an abelian von Neumann algebra. A derivation from $A$ to $\mathfrak{B}(\mathbb{H})$ is a linear map $\Delta$ satisfying $\Delta(AB) = (\Delta A)B + A(\Delta B)$. It is well known that every bounded derivation from $\mathcal{E}$ into $\mathfrak{B}(\mathbb{H})$ can be extended.
to an (inner) derivation of $\mathfrak{B}(\mathcal{K})$. Here we give a slight strengthening of this using a technique developed in [1] and [3].

**Theorem 4.1.** Let $\mathfrak{P}$ be the set of projections in an abelian von Neumann algebra $\mathfrak{A}$. Suppose $\Delta$ is a map of $\mathfrak{P}$ into $\mathfrak{B}(\mathcal{K})$ satisfying

(i) $\Delta(P + Q) = \Delta P + \Delta Q$ when $PQ = 0$,

(ii) $\Delta(PQ) = (\Delta P)Q + P(\Delta Q)$ and

(iii) $\|\Delta P\| < M$ for all $P, Q$ in $\mathfrak{P}$.

Then there is a $T$ in $\mathfrak{B}(\mathcal{K})$ with $\|T\| < 2M$ such that $\Delta = \Delta_T | \mathfrak{P}$.

**Proof.** Let $\mathfrak{E} = \{P_j, 1 \leq j \leq n\}$ be a finite subset of $\mathfrak{P}$ with $\sum_{j=1}^n P_j = I$. Define $T_\mathfrak{E} = \sum_{j=1}^n P_j \Delta P_j$. The standard argument shows that $\Delta I = 0$ and thus $\sum_{i=1}^n \Delta P_i = 0$.

Compute

$$T_\mathfrak{E} P_m - P_m T_\mathfrak{E} = \sum_{i \neq j} P_j (\Delta P_j) P_m - P_m \sum_{i \neq j} P_j \Delta P_i$$

$$= - \sum_{i \neq j} (\Delta P_j) P_i P_m - P_m \sum_{i \neq j} \Delta P_i$$

$$= - \left( \sum_{j \neq m} \Delta P_j \right) P_m - P_m \left( \sum_{i \neq m} \Delta P_i \right)$$

$$= (\Delta P_m) P_m + P_m (\Delta P_m) = \Delta P_m.$$  

Thus, for every projection $P$ in $\mathfrak{E}$, $T_\mathfrak{E} P - P T_\mathfrak{E} = \Delta P$. By Lemma 3.1,

$$\text{dist}(T_\mathfrak{E}, \mathfrak{E}) \leq 2 \sup_{P \in \mathfrak{E}''} \|T_\mathfrak{E} P - P T_\mathfrak{E}\| \leq 2 \sup_{P \in \mathfrak{E}''} \|\Delta P\| < 2M.$$  

Choose an $A_\mathfrak{E}$ in $\mathfrak{E}'$ with $\|T_\mathfrak{E} - A_\mathfrak{E}\| < 2M + 1/n$. Set $S_\mathfrak{E} = T_\mathfrak{E} - A_\mathfrak{E}$. Then $S_\mathfrak{E} P - P S_\mathfrak{E} = \Delta P$ for every $P$ in $\mathfrak{E}$.

Since all finite subsets of $\mathfrak{P}$ with sum $I$ form a directed set and the ball of radius $2M + 1$ is weakly compact, the net $\{S_\mathfrak{E}\}$ has a limit point $T$. Clearly, $\|T\| < 2M$ and $TP - PT = \Delta P$ for all $P$ in $\mathfrak{P}$.

It has come to our attention that Theorem 2.1 and Lemma 3.1 have been proven independently by Gilfeather and Larson [4], using similar methods.

**Added in Proof.** The author is greatly indebted to the referee for many improvements made on the manuscript.

**References**


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