PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 86, Number 2, October 1982

NONDISCRIMINATING SETS FOR H^∞

DONALD E. MARSHALL,[1] ARNE STRAY AND CARL SUNDBERG

ABSTRACT. We characterize the subsets S of the unit disk D with the property that any function defined on S that has a bounded harmonic extension to D, must also have a bounded analytic extension to D.

Let D denote the open unit disk in the complex plane, $\{z: |z| < 1\}$. Let h^∞ denote the space of bounded harmonic functions on D and let H^∞ denote the subspace consisting of the bounded analytic functions on D. We wish to characterize the subsets S of D with the property that the restriction spaces $H^\infty|_S$ and $h^\infty|_S$ coincide. Each $f \in h^\infty$ has a nontangential limit $f^*(e^{i\theta})$ almost everywhere with respect to Lebesgue measure on the boundary of the unit disk ∂D, and f is the Poisson integral of f^*. By identifying each $f \in h^\infty$ with its boundary value function f^*, h^∞ is isometrically isomorphic to L^∞, the essentially bounded measurable functions on ∂D. If B is a subspace of L^∞, we let $\hat{B}$ denote the space of harmonic extensions of elements of B to D. In keeping with standard practice, we denote the space of boundary functions of elements of H^∞ by H^∞ also, i.e. $H^\infty = \hat{H}^\infty$. If $H^\infty|_S \neq h^\infty|_S$ then S discriminates between H^∞ and all other Douglas algebras. In other words, if $H^\infty|_S \neq h^\infty|_S$ then $H^\infty|_S \neq \hat{B}|_S$ where B is any closed algebra such that $H^\infty \subsetneqq B \subseteq L^\infty$. Indeed, every such algebra B contains C, the space of continuous functions in L^∞. If $\hat{B}|_S = H^\infty|_S$ then by the Baire category theorem, there is a constant $M < \infty$ so that if $f \in \hat{B}$ then there is an $h \in \hat{H}^\infty$ with $f|_S = h|_S$ and $\|h\|_\infty \leq M\|f\|_\infty$. If $g \in h^\infty$, then the Cesàro means, $\sigma_n(g)$, are continuous and so $\sigma_n(g)|_S = h_n|_S$ for some $h_n \in H^\infty$ with $\|h_n\|_\infty \leq M\|\sigma_n(g)\|_\infty \leq M\|g\|_\infty$. Since H^∞ is closed under normal convergence on D, if h is a (normal) cluster point of $\{h_n\}$, then $h \in H^\infty$ and $h|_S = g|_S$. For this reason, we say S is a nondiscriminating set for H^∞ if $H^\infty|_S = h^\infty|_S$.

We note that if $H^\infty|_S = h^\infty|_S$ then S must be a Blaschke sequence. For if S is not a Blaschke sequence, then the restriction map from H^∞ to $H^\infty|_S$ is one-to-one. By the closed graph theorem, there would exist a bounded projection from L^∞ onto H^∞, which is false [7]. A sequence S is called an interpolation sequence if $H^\infty|_S = l^\infty(S)$, the space of all bounded sequences of complex numbers. Clearly, if S is an interpolation sequence, then $H^\infty|_S = h^\infty|_S$. We will characterize nondiscriminating sets in terms of interpolation sequences.

A measure σ, defined on D, is called a Carleson measure if there is a constant $C < \infty$ such that for any $\epsilon > 0$ and any $\alpha_0 \in [0, 2\pi]$, if $Q = \{re^{i\alpha}: 1 - \epsilon < r < 1 \text{ and } \alpha_0 < \alpha < \alpha_0 + \epsilon\}$ then $\sigma(Q) \leq C\epsilon$. Let $\rho(z, w) = |(z - w)/(1 - \overline{w}z)|$ be the pseudohyperbolic metric on D. Carleson [1] proved that $\{z_n\}_{n=1}^\infty$ is an interpolation

Received by the editors August 21, 1981 and, in revised form, December 7, 1981.
1980 *Mathematics Subject Classification*. Primary 30D50, 31A05.
Key words and phrases. Interpolation, Carleson measures.
[1]The first author is supported in part by an NSF grant.

sequence if and only if $\inf_m \prod_{n=1;n\neq m}^{\infty} \rho(z_n, z_m) > 0$. Equivalently (see [5, p. 287]), $\{z_n\}_{n=1}^{\infty}$ is an interpolation sequence if and only if

(1) there is a $\delta > 0$ so that $\rho(z_n, z_m) \geq \delta > 0$ for all n, m with $n \neq m$, and

(2) the measure $\sum(1-|z_n|)\delta_{z_n}$ is a Carleson measure, where δ_{z_n} denotes point mass at z_n.

It is not hard to see that a sequence S is a finite union of interpolation sequences if and only if $\mu_S \equiv \sum_{z_n \in S}(1-|z_n|)\delta_{z_n}$ is a Carleson measure [9, p. 608]. If S is a Blaschke sequence, let B_S denote the canonical Blaschke product with zero set S and let $N_\delta(S) = \{z\colon \rho(z,S) < \delta\}$. By the work in [4], S is a finite union of interpolation sequences if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ so that for each $z_n \in S$, the component of $\{z\colon |B_S(z)| < \delta\}$ containing z_n is contained in $\{z\colon \rho(z, z_n) < \epsilon\}$.

THEOREM 1. *If $H^\infty|_S = h^\infty|_S$, then S is the union of a finite number of interpolation sequences.*

PROOF. By the Baire category theorem, there is a constant $C < \infty$ such that if $f \in h^\infty$ there exists $h \in H^\infty$ with $f|_S = h|_S$ and $\|h\|_\infty \leq C\|f\|_\infty$. If S is not a finite union of interpolation sequences, we can find linear fractional transformations τ_n of D onto D such that $B_S \circ \tau_n$ converges to zero, uniformly on compact subsets of D. For if $\tau_\lambda(z) = (z-\lambda)/(1-\overline{\lambda}z)$ and if $Q(\lambda) = \{re^{i\alpha}\colon 1-|\lambda| < r < 1 \text{ and } \arg\lambda < \theta < (\arg\lambda) + 1 - |\lambda|\}$ then

$$\begin{aligned}\sum_{k=1}^{\infty} 1 - |\tau_\lambda^{-1}(z_k)|^2 &= \sum_{k=1}^{\infty} \frac{(1-|z_k|^2)(1-|\lambda|^2)}{|1-\overline{\lambda}z_k|^2} \\ &\geq \sum_{z_k \in Q(\lambda)} \frac{1-|z_k|^2}{1-|\lambda|^2} \cdot \frac{1}{5} \\ &\geq \frac{1}{10}\frac{\mu_S(Q(\lambda))}{1-|\lambda|}.\end{aligned}$$

So if μ_S is not a Carleson measure we can choose λ_n so that $\lim_n \sum_k (1-|\tau_{\lambda_n}^{-1}(z_k)|^2) = \infty$. Since $\{\tau_{\lambda_n}^{-1}(z_k)\}_{k=1}^{\infty}$ are the zeros of $B_S \circ \tau_{\lambda_n}$, $B_S \circ \tau_{\lambda_n}$ converges to zero uniformly on compact subsets of Δ as $n \to \infty$.

Let B_n be a finite partial product of $B_S \circ \tau_n$ such that $\{B_n\}$ converges to zero uniformly on compact subsets of Δ. For each $f \in h^\infty$, let $S_n(f)$ be the unique element of least norm in H^∞ that agrees with f on the zeros of B_n. By the Baire category argument above, applied to $f \circ \tau_n$, we conclude $\|S_n(f)\|_\infty \leq C\|f\|_\infty$. Since the C-ball of H^∞ is compact in the weak-* topology, by the Tychonoff product theorem, the set of maps from the unit ball of h^∞ into the C-ball of H^∞ is compact under pointwise convergence. Hence the net $\{S_n\}$ has a subnet $\{S_{n_\alpha}\}$ with the property that $\{S_{n_\alpha}(f)\}$ is convergent, in the weak-* topology, for every f in the unit ball of h^∞. Since $S_{n_\alpha}(\lambda f) = \lambda S_{n_\alpha}(f)$ for $\lambda \in \mathbf{C}$, this subnet converges pointwise on all of h^∞. Let $S(f)$ denote the (weak-*) limit of $S_{n_\alpha}(f)$, for $f \in h^\infty$. Since $S_n(f+g) - S_n(f) - S_n(g)$ is bounded by $2C(\|f\|_\infty + \|g\|_\infty)$ and vanishes on the zeros of B_n, $S_{n_\alpha}(f+g) - S_{n_\alpha}(f) - S_{n_\alpha}(g)$ converges weak-* to 0. We conclude $S(f+g) = S(f) + S(g)$ and hence S is linear. Likewise if $f \in H^\infty$, $f - S_n(f)$ is bounded and vanishes on the zeros of B_n and hence $f = S(f)$. Hence S is a bounded linear projection of h^∞ onto H^∞, which is impossible. This proves μ_S is a Carleson measure, and the theorem follows.

REMARK. If S is the union of M interpolation sequences then for ϵ small enough, each component of $N_\epsilon(S)$ contains M or fewer elements of S.

\ THEOREM 2. *Let S be the union of a finite number of interpolation sequences. Then $H^\infty|_S = h^\infty|_S$ if and only if for ϵ sufficiently small, there is a function f analytic and bounded by 1 on $N_\epsilon(S)$ with $f(z) = \overline{z}$, for all $z \in S$.*

PROOF. Suppose such an f exists. Choose $\delta > 0$ such that if $z_n \in S$ then the component of $N_\delta(S)$ containing z_n is contained in $\{z\colon \rho(z, z_n) < \epsilon/2\}$. Let C_j be a component of $N_\delta(S)$ and let $\{w_1, \ldots, w_k\} = S \cap C_j$. Let $\varphi(z) = (z - w_k)/(1 - \overline{w}_k z)$ and suppose $h \in h^\infty$. Write $h(\varphi^{-1}(z)) = h_1(z) + h_2(\overline{z})$ where h_1 and h_2 belong to H^2 and $h_2(0) = 0$. A simple estimate of the Poisson kernel shows that if $|z| < \frac{1}{2}$, then $|h_j(z)| < 3\|h\|_\infty$. Now if $g(z) = (f(\varphi^{-1}(z)) - \overline{w}_k)/(1 - w_k f(\varphi^{-1}(z)))$ then $|g(z)| < \frac{1}{2}$ if $|z| < \epsilon/2$ by Schwarz's lemma, and $g(\varphi(w_i)) = \overline{\varphi(w_i)}$ for $1 \le i \le k$. Let $H_j(z) = h_1(\varphi(z)) + h_2(g(\varphi(z)))$. Then H_j is analytic on $N_\delta(S)$, $|H_j(z)| < 6\|h\|_\infty$ and $H_j(w_i) = h(w_i)$ for $1 \le i \le k$. Since $\{z\colon |B_S(z)| < \eta\} \subset N_\delta(S)$ for small η, we may define H on $\{z\colon |B_S(z)| < \eta\}$ by $H(z) = H_j(z)$ if $z \in C_j$. By a theorem of Carleson (see e.g. [3, p. 203]) there is a function $K \in H^\infty$ with $K|_S = H|_S = h|_S$.

Conversely, suppose $H^\infty|_S = h^\infty|_S$. As before there is an $M < \infty$ such that for each $h \in h^\infty$, there is an $f \in H^\infty$ with $f|_S = h|_S$ and $\|f\|_\infty \le M\|h\|_\infty$. By hypothesis, S is a finite union of interpolation sequences. So that for ϵ sufficiently small, if $z_n \in S$, then $\{z\colon \rho(z, z_n) < 1/M\}$ contains the component of $N_\epsilon(S)$ to which z_n belongs. For each component C_j of $N_\epsilon(S)$, choose one $z_{n_j} \in S \cap C_j$ and $f_j \in H^\infty$ with $f_j|_S = ((\overline{z} - \overline{z}_{n_j})/(1 - z_{n_j}\overline{z}))|_S$ and $\|f_j\|_\infty \le M$. By Schwarz's lemma, $|f_j(z)| \le M\rho(z, z_{n_j})$ and so $|f_j(z)| < 1$ on C_j. Define f on $N_\epsilon(S)$ by

$$f(z) = \frac{f_j(z) + \overline{z}_{n_j}}{1 + z_{n_j} f_j(z)}$$

for $z \in C_j$, $j = 1, 2, \ldots$. Then f is analytic on $N_\epsilon(S)$, $|f(z)| < 1$ on $N_\epsilon(S)$ and $f|_S = \overline{z}|_S$.

Since $\overline{z} = z$ when z is real, the following corollary obtains, by Theorems 1 and 2.

COROLLARY 1. *If S is contained in the interval $(-1, 1)$ then $H^\infty|_S = h^\infty|_S$ if and only if μ_S is a Carleson measure.*

COROLLARY 2. *If S is the union of two interpolation sequences then $H^\infty|_S = h^\infty|_S$.*

PROOF. Choose $\delta > 0$ so that each component of $N_\delta(S)$ contains at most two elements of S. If z_1 and z_2 are any two points in D then $\rho(\overline{z}_1, \overline{z}_2) = \rho(z_1, z_2)$. By Pick's theorem, there is an $f \in H^\infty$ with $\|f\|_\infty \le 1$ and $f(z_j) = \overline{z}_j$ for $j = 1, 2$. By Theorem 2, $H^\infty|_S = h^\infty|_S$.

COROLLARY 3. *If $H^\infty|_S = h^\infty|_S$ then there is a bounded linear operator T from h^∞ into H^∞ such that $(Tf)|_S = f|_S$ for all $f \in h^\infty$.*

PROOF. By Theorem 1, S is a finite union of interpolation sequences. In [2], there is constructed a projection P of H^∞ into H^∞ such that $(Ph)|_S = h|_S$ and if $g|_S = h|_S$ then $Ph = Pg$. Since $H^\infty|_S = h^\infty|_S$, there is a constant $M < \infty$ so that for each $f \in h^\infty$, there is an $h \in H^\infty$ with $h|_S = f|_S$ and $\|h\|_\infty \le M\|f\|_\infty$. Define $Tf = Ph$. It is easy to verify that this defines the desired operator.

To find a geometric condition equivalent to $H^\infty|_S = h^\infty|_S$ when S is a finite union of interpolation sequences, we note the following. If μ_S is a Carleson measure, then $H^\infty|_S = h^\infty|_S$ if and only if for ϵ sufficiently small, if E is a component of $N_\epsilon(S)$, if $z_n \in S \cap C$ and if $E = \{(z - z_n)/(1 - z_n z)\colon z \in S \cap C\}$ then the interpolation problem

$$(1) \qquad f \in H^\infty, \quad \|f\|_\infty < 1/\epsilon, \qquad f(z) = \overline{z} \quad \text{for } z \in E$$

has a solution. Indeed if $H^\infty|_S = h^\infty|_S$, let $f \in H^\infty$ with

$$f|_S = (\overline{z} - \overline{z}_n)/(1 - z_n\overline{z})|_S$$

and $\|f\|_\infty \le 1/\epsilon$. Then $g(z) = f((z+z_n)/(1+\overline{z}_n z))$ solves the interpolation problem. The converse has a proof very similar to the proof of Theorem 2. If the interpolation problem (1) always has a solution, choose $\delta > 0$ so that if $z_n \in S$ then $\{z\colon \rho(z, z_n) < \epsilon\}$ contains the component of $N_\delta(S)$ to which z_n belongs. For each component C_j of $N_\delta(S)$ choose $z_{n_j} \in C_j \cap S$ and $f_j \in H^\infty$ with $\|f_j\|_\infty \le 1/\epsilon$ and $f_j(z) = \overline{z}$ for $z \in \{(z - z_{n_j})/(1 - \overline{z}_{n_j} z)\colon z \in S \cap C_j\}$ (a possibly smaller set than that required in the interpolation problem (1)). By Schwarz's lemma, $|f_j(z)| < 1$ on C_j. Let

$$f(z) = \frac{f_j\left(\dfrac{z - z_{n_j}}{1 - \overline{z}_{n_j} z}\right) + \overline{z}_{n_j}}{1 + z_{n_j} f\left(\dfrac{z - z_{n_j}}{1 - \overline{z}_{n_j} z}\right)}$$

for $z \in C_j$. Then f is analytic on $N_\delta(S)$, $f|_S = \overline{z}|_S$ and $|f(z)| < 1$ on $N_\delta(S)$. By Theorem 2, $H^\infty|_S = h^\infty|_S$. The Pick-Nevanlinna interpolation theorem (see e.g. [8]) gives a geometric condition equivalent to the solution of the interpolation problem (1). Alternatively, we deduce the following corollary.

COROLLARY 4. *$H^\infty|_S = h^\infty|_S$ if and only if μ_S is a Carleson measure and there is an $\epsilon > 0$ such that if $w_1, \dots, w_k$ are in S and belong to the same component of $N_\epsilon(S)$ and if $\alpha_i = (w_i - w_k)/(1 - \overline{w}_k w_i)$, the Lagrange interpolating function*

$$f(z) = \sum_{i=1}^{k} \overline{\alpha}_i \prod_{\substack{j=1 \\ j \ne i}}^{k} \frac{z - \alpha_j}{\alpha_i - \alpha_j} = C_1 z + \cdots + C_{k-1} z^{k-1}$$

satisfies $|C_j| \le 1/\epsilon$ for $j = 1, \dots, k-1$.

Corollary 4 follows easily from the above remarks and the following lemma.

LEMMA. *For each positive integer N, there is a $\delta_N > 0$ such that for any $E = \{z_1, \dots, z_N\}$ contained in $\{z\colon |z| < \frac{1}{2}\}$ and any polynomial $Q(z) = C_0 + C_1 z + \cdots + C_{N-1} z^{N-1}$ we have*

$$\inf\{\|Q - B_E h\|_\infty \colon h \in H^\infty\} \ge \delta_N \max_{0 \le j \le N-1} |C_j|.$$

PROOF OF THE LEMMA. If not, there are polynomials $Q_k(z) = C_0^{(k)} + C_1^{(k)} z + \cdots + C_{N-1}^{(k)} z^{N-1}$ with $\max_{0 \le j \le N-1} |C_j^{(k)}| = 1$ and sets $E_k = \{z_1^{(k)}, \dots, z_N^{(k)}\}$ and $h_k \in H^\infty$ such that $\lim_{k\to\infty} \|Q_k - B_{E_k} h_k\|_\infty = 0$. By taking subsequences, we may suppose there is a polynomial Q of degree at most $N-1$, a Blaschke product B of degree N and an $h \in H^\infty$ such that Q_k converges to Q, B_{E_k} converges to B

and h_k converges to h, uniformly on compact subsets of D. Thus $Q = Bh$ which is impossible. This proves the lemma.

Chapters 10 and 11 of [6], for example, give various techniques for computing the Lagrange interpolating function. Since we only need to prove that the coefficients are bounded by a constant which can be allowed to depend on the number of interpolation points k, an efficient test can be based on Aitken's lemma [6, p. 204] as follows. Let f_n be the unique polynomial of degree at most $n-1$ such that $f_n(w_i) = \overline{w}_i$, $1 \le i \le n$. Let $A_n(w_1, \dots, w_n)$ be the coefficient of z^{n-1} in f_n. Clearly

$$f_n(z) = f_{n-1}(z) + A_n(w_1, \dots, w_n) \prod_{i=1}^{n-1} (z - w_i).$$

Since $|w_i| < 1$, by induction the coefficients of $f_1, \dots, f_k$ are bounded by a constant (depending on k) if and only if $\max\{|A_1(w_1)|, \dots, |A_k(w_1, \dots, w_k)|\}$ is bounded by a constant depending only on k. The coefficients A_k can be computed recursively as follows. Clearly $A_1(w_i) = \overline{w}_i$. The reader can easily prove by induction, or deduce from Aitken's lemma, that

$$A_n(w_1, \dots, w_n) = \frac{A_{n-1}(w_1, \dots, w_{n-2}, w_n) - A_{n-1}(w_1, \dots, w_{n-1})}{w_n - w_{n-1}}.$$

An efficient computational scheme to compute $A_1(w_1), \dots, A_k(w_1, \dots, w_k)$ requires only $k(k-1)/2$ divisions.

The analytic functions are the harmonic functions which satisfy the Cauchy-Riemann equations. This motivates the next corollary. It says that if $H^\infty|_S = h^\infty|_S$ then for δ sufficiently small, the elements of S in each component of $N_\delta(S)$ are almost on a straight line.

COROLLARY 5. *Suppose $H^\infty|_S = h^\infty|_S$. If $\{x_n, y_n, z_n\}_{n=1}^\infty$ is any sequence of triples from S with $\rho(x_n, y_n) \to 0$ and $\rho(y_n, z_n) \to 0$ then*

$$\arg\left(\frac{z_n - y_n}{x_n - y_n}\right)^2 \to 0$$

as well. Here arg *denotes the principal determination $-\pi < \arg\theta \le \pi$.*

PROOF. Write $x_n = y_n + \epsilon_n$, $z_n = y_n + \delta_n$, $\alpha_n = (x_n - y_n)/(1 - \overline{y}_n x_n) = \epsilon_n/(1 - |y_n|^2 - \epsilon_n \overline{y}_n)$, and $\beta_n = (z_n - y_n)/(1 - \overline{y}_n z_n) = \delta_n/(1 - |y_n|^2 - \delta_n \overline{y}_n)$. By the above remarks

$$A_3(0, \alpha_n, \beta_n) = \left(\frac{\overline{\alpha}_n}{\alpha_n} - \frac{\overline{\beta}_n}{\beta_n}\right)/(\alpha_n - \beta_n)$$

is bounded. Since $\alpha_n \to 0$ and $\beta_n \to 0$ and

$$\frac{\overline{\alpha}_n}{\alpha_n} - \frac{\overline{\beta}_n}{\beta_n} = \frac{\overline{\epsilon}_n}{\epsilon_n}\left(1 + \frac{\alpha_n}{\epsilon_n}(\overline{\epsilon}_n y_n - \epsilon_n \overline{y}_n)\right) - \frac{\overline{\delta}_n}{\delta_n}\left(1 + \frac{\beta_n}{\delta_n}(\overline{\delta}_n y_n - \delta_n \overline{y}_n)\right),$$

we must have $\overline{\epsilon}_n/\epsilon_n - \overline{\delta}_n/\delta_n \to 0$. Thus $\arg((z_n - y_n)/(x_n - y_n))^2 \to 0$.

For example if $\{x_n\}$ is an interpolation sequence contained in $(-1, 1)$ and if $S = \{x_n + i\epsilon_n, x_n + \epsilon_n, x_n \colon n = 1, 2, 3, \dots\}$ then if ϵ_n converges to zero sufficiently fast, S is the union of three interpolation sequences, but $H^\infty|_S \ne L^\infty|_S$.

REFERENCES

1. L. Carleson, *An interpolation theorem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.
2. P. G. Casazza, R. W. Pengra and C. Sundberg, *Complemented ideals in the disk algebra*, Israel J. Math. **37** (1980), 76–83.
3. P. L. Duren, *Theory of H^p spaces*, Academic Press, New York and London, 1970.
4. J. P. Earl, *On the interpolation of bounded sequences by bounded functions*, J. London Math. Soc. **2** (1970), 544–548.
5. J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
6. P. Henrici, *Elements of numerical analysis*, Wiley, New York, 1964.
7. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
8. D. E. Marshall, *An elementary proof of the Pick-Nevanlinna interpolation theorem*, Michigan Math. J. **21** (1974), 219–223.
9. G. McDonald and C. Sundberg, *Toeplitz operators on the disc*, Indiana Univ. Math. J. **28** (1979), 595–611.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195

DEPARTMENT OF MATHEMATICS, AGDER DISTRIKTSHOGSKOLE, KRISTIANSAND, NORWAY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916