NUCLEAR FACES OF STATE SPACES OF C*-ALGEBRAS

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Abstract. Let $B_1$ and $B_2$ be maximal abelian subalgebras of C*-algebras $A_1$ and $A_2$, and suppose that for each pure state $\psi_1$ of $B_1$, the von Neumann algebra $p_{\psi_1}A_1^*p_{\psi_1}$ is injective, where $p_{\psi_1}$ is the common support in $A_1'^*$ of all the states of $A_1$ which extend $\psi_1$. Then $B_1 \otimes B_2$ is a maximal abelian subalgebra of any C*-tensor product $A_1 \otimes A_2$.

1. Introduction. Let $A_1$ and $A_2$ be C*-algebras with maximal abelian C*-subalgebras (masas) $B_1$ and $B_2$ respectively. The unique C*-tensor product $B = B_1 \otimes B_2$ of $B_1$ and $B_2$ may be regarded as a C*-subalgebra of any C*-tensor product $A_1 \otimes A_2$. Wassermann [18] showed that $B$ is a masa in the minimal (spatial) tensor product $A_1 \otimes_{\text{min}} A_2$, and asked whether $B$ is a masa in all the tensor products. The corresponding result for von Neumann algebras is an easy consequence of the Commutation Theorem [16, p. 229]. The answer to Wassermann's question is trivially affirmative if $A_1$ (or $A_2$) is nuclear (equivalently $A_1'^*$ (or $A_2'^*$) is injective). The purpose of this note is to show that the answer remains affirmative even if the assumption of nuclearity is replaced by a much weaker "nuclear extension property" (NEP) of the subalgebra $B_1$, which requires only the injectivity of certain small parts of $A_1'^*$. The (NEP) is also much weaker than the "extension property" (EP) of $B_1$—the property that pure states of $B_1$ (including the zero functional) have unique extensions to pure state of $A_1$ [1, 2, 4, 5]. Indeed the (NEP) holds if no pure state of $B_1$ has two distinct equivalent pure state extensions to $A_1$, a property which it is natural to call the "simplex extension property" (SEP) because of its connections with the geometry of the state space of $A_1$ [6, 7, 8].

2. Nuclear faces. For a C*-algebra $A$, let $Q(A)$ be the quasi-state space of $A$ in the weak* topology:

$$Q(A) = \{ \phi \in A^*: \phi > 0, \|\phi\| < 1 \}. $$

For a closed face $F$ of $Q(A)$, there is a unique closed projection $p^F$ in $A'^*$ such that

$$F = \{ \phi \in Q(A): \phi(1 - p^F) = 0 \}$$

[15, 3.11.10]. The linear span of $F$ in $A^*$ is the predual of the von Neumann algebra $p^F A'^* p^F$. For $\phi$ in $F$, let $(\mathcal{K}_\phi, \pi_\phi, \xi_\phi)$ be the corresponding cyclic representation of $A'^*$, and let $p_\phi$ be the orthogonal projection of $\mathcal{K}_\phi$ onto the linear subspace of all

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vectors $\eta$ in $\mathcal{H}_\phi$ for which the vector functional $a \to \langle \pi_\phi(a) \eta, \eta \rangle$ lies in the cone generated by $F$. Then $p_\phi^F = \pi_\phi(p^F)$, and the following conditions have been shown to be equivalent [6, 7]:

(A1) $p^F A^{**} p^F$ is abelian.

(A2) $F$ is a Choquet simplex.

(A3) For each $\phi$ in $F$, $p_\phi^F \pi_\phi(A)^{**} p_\phi^F$ is abelian.

(A4) For each $\phi$ in $F$, $\pi_\phi(A)'$ is abelian.

(A5) Distinct pure states in $F$ are inequivalent.

If these conditions are satisfied, $F$ is said to be abelian.

Various classes of faces can be defined by replacing abelianess in (A1) by other algebraic properties of the von Neumann algebra $p^F A^{**} p^F$, for example its type (in the sense of Murray and von Neumann). Another possibility is injectivity, and for this it is easy to obtain analogues of conditions (A2)–(A4). An affine mapping of $F$ into $F$ is a morphism if it has a completely positive linear extension to the subspace of $A^*$ spanned by $F$, and it takes states into states.

**Proposition 1.** The following conditions on a closed face $F$ of $Q(A)$ are equivalent:

(N1) $p^F A^{**} p^F$ is injective.

(N2) There is a net of morphisms of $F$ of finite rank converging to the identity on $F$ in the point-norm topology.

(N3) For each $\phi$ in $F$, $p_\phi^F \pi_\phi(A)^{**} p_\phi^F$ is injective.

(N4) For each $\phi$ in $F$, $\pi_\phi(A)'$ is injective.

(N4)' For each $\phi$ in $F$, $\pi_\phi(A)'$ is injective.

**Proof.** (N1) $\Leftrightarrow$ (N2). The linear span of $F$ is identified with the predual of $p^F A^{**} p^F$. Condition (N2) is then equivalent to the semidiscreteness of $p^F A^{**} p^F$ [13], and hence to injectivity [11] (see also [12, 19]).

(N1) $\Rightarrow$ (N3). $p_\phi^F \pi_\phi(A)^{**} p_\phi^F = \pi_\phi(p^F A^{**} p^F)$, which is a direct summand of $p^F A^{**} p^F$.

(N3) $\Rightarrow$ (N1). If $\{\phi_\lambda : \lambda \in \Lambda\}$ is a maximal family of states in $F$ with orthogonal central supports, then $p^F A^{**} p^F \cong \bigoplus p_\phi^F \pi_\phi(A)^{**} p_\phi^F$, and so $p^F A^{**} p^F$ is injective.

(N3) $\Rightarrow$ (N4). The commutant of an injective von Neumann algebra is injective [17], so $\pi_\phi(A)' p_\phi^F$ is injective. But the central support of $p_\phi^F$ in $\pi_\phi(A)^{**}$ is the identity, so $\pi_\phi(A)'$ is isomorphic to $\pi_\phi(A)' p_\phi^F$.

(N4) $\Rightarrow$ (N4'). This is immediate from the result of [17].

(N4') $\Rightarrow$ (N3). This is immediate.

**Definition 2.** A closed face $F$ of $Q(A)$ is nuclear if the equivalent conditions (N1)–(N4)' hold.

The remarkable characterisation of nuclear $C^*$-algebras obtained by Choi and Effros [9, 10] from the results of Connes [11] can now be phrased as

$A$ is nuclear $\iff Q(A)$ is nuclear.

3. **The NEP and tensor products.** It is now possible to study nuclearity of the face associated with a single pure state of a $C^*$-subalgebra $B$ of $A$. Thus if $\psi$ is an extreme point of $Q(B)$ (either a pure state or the zero functional), let

$$Q_\psi(A) = \{ \phi \in Q(A) : \phi |_B = \psi \}.$$
Then $Q(A)$ is a nonempty closed face of $Q(A)$. The $C^*$-subalgebra $B$ is said to have the extension property (EP) in $A$ if $Q(B)$ contains only one functional, for each extreme point $\psi$ of $Q(B)$ [1, 4, 5].

**Definition 3.** A $C^*$-subalgebra $B$ of $A$ has the simplex (respectively, nuclear) extension property (SEP (respectively, NEP)) in $A$, if for each extreme point $\psi$ of $Q(B)$, the face $Q_{\psi}(A)$ of $Q(A)$ is abelian (respectively, nuclear).

Clearly (EP) $\Rightarrow$ (SEP) $\Rightarrow$ (NEP). Abelian $C^*$-subalgebras with the (EP) have been extensively studied in [1, 4, 5]; nonatomic masas in type I factors do not have the (SEP) [14, 1], but it does not seem to be easy to find masas which do not have the (NEP).

The connection between the (NEP) and tensor products is established in the following lemma.

**Lemma 4.** Let $A_1 \otimes A_2$ be a $C^*$-tensor product of $C^*$-algebras $A_1$ and $A_2$, let $\psi: A_1 \otimes A_2 \to A_1 \otimes_{\min} A_2$ be the canonical *-homomorphism, and let $\phi$ be a state of $A_1 \otimes A_2$ whose restriction to $A_1$ lies in a nuclear face of $Q(A_1)$. Then there is a state $\phi_{\min}$ of $A_1 \otimes_{\min} A_2$ such that $\phi = \phi_{\min} \circ \psi$.

**Proof.** There is a faithful embedding $\Phi$ of the algebraic tensor product $A_1^* \otimes A_2^*$ in $(A_1 \otimes A_2)^*$ which is normal in each variable separately, and acts as the identity on $A_1 \otimes A_2$ (see the proof of [3, Theorem 2]). Let $\psi = \phi \circ \Phi$. The restriction of $\phi$ to $A_1$ is given by $a_1 \to \langle \phi(a_1 \otimes 1) \rangle$, so that there is a (closed) projection $p$ in $A_1^*$ with $\psi(p \otimes 1) = 1$ such that $pA_1^*p$ is injective, hence semidiscrete [11]. Now $\psi$ is a normalised positive linear functional on $A_1^* \otimes A_2^*$ which is normal in each variable separately, so $\psi$ is continuous and of norm 1 for the minimal $C^*$-norm on $pA_1^*p \otimes A_2^*$ [13, Theorem 4.1]. Hence

$$|\phi(x)| = |\psi((p \otimes 1)x(p \otimes 1))| \leq \|p \otimes 1\| \|x\|_{\min}$$

Thus $\phi$ factors through $A_1 \otimes_{\min} A_2$.

Minor modifications of the proof of Lemma 4 show that if the restriction of $\phi$ to $A_1$ lies in an abelian face of $Q(A_1)$, then $\phi$ factors through the tensor product $A_1 \otimes_{\lambda} A_2$ of $A_1$ and $A_2$ in the least cross-norm of Banach spaces. Hence if $B_1$ is an abelian $C^*$-subalgebra with the (SEP) in $A_1$, and $B_2$ is a $C^*$-subalgebra with the (SEP) in $A_2$, then $B_1 \otimes B_2$ has the (SEP) in $A_1 \otimes_{\min} A_2$.

It is very easy to see that if $B_1$ is an abelian $C^*$-subalgebra with the (EP) in $A_1$, and $B_2$ is any $C^*$-subalgebra with the (EP), (SEP) or (NEP) in $A_2$, then $B_1 \otimes B_2$ has the (EP), (SEP) or (NEP) respectively in $A_1 \otimes_{\min} A_2$. Hence if $B_1$ and $B_2$ are masas with the (EP), then $B_1 \otimes B_2$ is a masa. The following result generalises this.

**Theorem 5.** Let $A = A_1 \otimes A_2$ be any $C^*$-tensor product of $C^*$-algebras $A_1$ and $A_2$, let $B_j$ be a masa of $A_j$ ($j = 1, 2$), and $B = B_1 \otimes B_2$. Suppose that $B_1$ has the (NEP) in $A_1$. Then $B$ is a masa in $A$.

**Proof.** Let $C$ be an abelian $C^*$-subalgebra of $A$ containing $B$, and $\Psi: A \to A_1 \otimes_{\min} A_2$ be the canonical *-homomorphism, so that $\Psi$ is the identity on $B$. Then
ψ(C) is an abelian C*-subalgebra of $A_1 \otimes_{\min} A_2$ containing $B$, so $\Psi(C) = B$ [18, Corollary 6].

Let $\phi$ and $\phi'$ be multiplicative linear functionals on $C$ which coincide on $B$, and $\psi$ and $\psi'$ be norm-preserving extensions of $\phi$ and $\phi'$ to functionals in $Q(A)$. Since the restrictions of $\phi$ and $\phi'$ to $B$ are multiplicative, it follows from Lemma 4 that there are functionals $\phi_{\min}$ and $\phi'_{\min}$ in $Q(A_1 \otimes_{\min} A_2)$ such that $\phi = \phi_{\min} \circ \psi$, $\phi' = \phi'_{\min} \circ \psi$. Then $\phi_{\min}$ and $\phi'_{\min}$ coincide on $B = \Psi(C)$, so $\phi$ and $\phi'$ coincide on $C$, and $\psi = \psi'$. It follows from the Stone-Weierstrass Theorem that $B = C$, so $B$ is a masa in $A$.

The question whether Theorem 5 is valid without the assumption of the (NEP) remains open.

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