ON PROXIMALITY IN $L_1(T \times S)$

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Abstract. It is proved that if $G$ and $H$ are finite-dimensional subspaces of $L_1(S)$ and $L_1(T)$ respectively then each element of $L_1(T \times S)$ has a best approximation in the subspace $L_1(T) \otimes G + H \otimes L_1(S)$.

1. Introduction. Let $W$ be a subspace of a normed linear space $X$. $W$ is said to be proximinal in $X$ if to each $f$ in $X$ there corresponds a closest point $w^*$ in $W$; that is, a point $w^*$ in $W$ such that $\| f - w^* \| \leq \| f - w \|$ for all $w$ in $W$.

We consider two finite measure spaces $(T, \Theta, \mu)$ and $(S, \Phi, \nu)$. The product space $T \times S$ becomes a measure space $(T \times S, \Omega, \sigma)$ by means of a standard construction. Let $G = [g_1, g_2, \ldots, g_n]$ be a finite-dimensional subspace of $L_1(S)$ and $H = [h_1, h_2, \ldots, h_m]$ be a similar subspace of $L_1(T)$. Set $U = L_1(T) \otimes G$ and $V = H \otimes L_1(S)$. A typical element $u$ of $U$ has the form $u(t, s) = \sum x_i(t)g_i(s)$ where $x_i \in L_1(T)$. We shall take $X$ to be $L_1(T \times S)$ and $W$ to be $U + V$.

It is known from [3] and earlier work in [1] that if $f$ is essentially bounded on $T \times S$, then it has a closest point in $W$ (distance being measured in the $L_1$-norm). We shall establish the more general result.

Theorem. The subspace $W = L_1(T) \otimes G + H \otimes L_1(S)$ is proximinal in $L_1(T \times S)$.

2. Preliminaries. In this section we present the three strands which will combine to prove the main result.

Unadorned norm symbols will denote the $L_1$-norm on $T \times S$, whereas subscripts will be used to denote $L_1$-norms on $T$ and $S$. For example,

$$\|f\| = \int_{T \times S} |f(t, s)|d\mu d\nu, \quad f \in L_1(T \times S),$$

while

$$\|v\|_S = \int_S |v(s)|d\nu, \quad v \in L_1(S).$$

The first strand is the Dunford-Pettis theorem [2, p. 294].

Theorem A (Dunford-Pettis). A set $K$ in $L_1(T \times S)$ is weakly relatively sequentially compact if and only if it is bounded and

$$\lim_{\sigma(E) \to 0} \int_E f d\sigma = 0 \quad \text{uniformly for } f \text{ in } K.$$
By the Eberlein-Smulian theorem [2, p. 430], this condition is also necessary and sufficient for weak relative compactness in $L_1(T \times S)$. The sufficiency in Theorem A only holds good since $(T \times S, \Omega, \sigma)$ is a finite measure space.

The second result comes from [3]. It is a summary of the construction carried out in the proof of Theorem 1 therein. We adopt the notation $f_t, f^s$ where $f_t(s) = f(t, s) = f^s(t)$. By the Fubini theorem, if $f \in L_1(T \times S)$, then $f_t \in L_1(S)$ for almost all $t$ in $T$ and $f^s \in L_1(T)$ for almost all $s$ in $S$.

**Lemma B.** To each $f$ in $L_1(T \times S)$ there corresponds a closest point $u$ in $U$ such that $u_t$ is a closest point in $G$ to $f_t$ for almost all $t$ in $T$.

Finally, our third tool is the following elementary result:

**Lemma C.** There exists a function $g$ in $L_1(S)$ such that, for each $u$ in $U$,

(i) $|u(t, s)| \leq g(s)\|u_t\|_S$,

(ii) $\|u_t\|_T \leq g(s)\|u\|_T$,

for almost all $t$ in $T$ and $s$ in $S$.

**Proof.** Set $d_j^{-1} = \inf_{c_j \in R} \|\sum c_i g_i + g_j\|_S$. Since the $g_i$ are linearly independent, we have $d_j^{-1} > 0$ for $j = 1, 2, \ldots, n$. Let $u = \sum x_i g_i$ in $U$ and let $T_j = \{t \in T: x_j(t) \neq 0\}$. Then for $t$ in $T_j$ we have

$$\left|\sum x_i(t) g_i\right|_S = |x_j(t)| \sum_{i=1}^n \frac{x_i(t)}{x_j(t)} g_i \geq |x_j(t)| d_j^{-1}.$$

So for all $t$ in $T$, $|x_j(t)| \leq d_j \|u_t\|_S$. Now

$$|u(t, s)| \leq \sum_i |x_i(t)| |g_i(s)| \leq \|u_t\|_S \sum_i d_j |g_i(s)|.$$

Choosing $g = \sum_i d_j |g_i|$, (i) is proved. To obtain (ii),

$$\|u_t\|_T = \int_T |u(t, s)| d\mu \leq \int |g(s)| \|u_t\|_S d\mu = g(s)\|u\|.$$
for almost all $t$ in $T$. By Lemma C(i)

$$|B_U \nu(t, s)| \leq g(s)\|B_U \nu\|_S \leq 2g(s)(\|f\|_S + \|v\|_S).$$

Now applying Lemma C(ii) to $V$ instead of $U$, there is an $h$ in $L_1(T)$ such that $\|v\|_S \leq h(t)\|v\|$ for all $v$ in $V$. Then

$$|B_U \nu(t, s)| \leq 2g(s)(\|f\|_S + h(t)\|v\|) \leq 2g(s)(\|f\|_S + k(h(t)))$$

for $v$ in $K$.

The right-hand side of this inequality is a member of $L_1(T \times S)$ which is independent of $v$ in $K$. Hence if $Q$ is a measurable set in $T \times S$,

$$\int_Q |B_U \nu| d\sigma \to 0 \text{ as } \sigma(Q) \to 0 \text{ uniformly over } v \text{ in } K.$$

By the Dunford-Pettis theorem (Theorem A), $B_U K$ is weakly relatively compact.

Theorem D is the essential tool used to establish the proximinality of $W = U + V$ in $L_1(T \times S)$. However, a necessary condition for $W$ to be proximinal is that it be closed. We need to use the fact that $W$ is closed. This result was given in [3] and we reproduce it here on account of its brevity.

**Lemma E.** The subspace $W = U + V$ is closed in $L_1(T \times S)$. There is a constant $\beta$ such that each element $w$ of $W$ has a representation $w = u + v$ with $u \in U, v \in V$ and $\|u\| + \|v\| \leq \beta \|w\|$.

**Proof.** Let biorthonormal bases $\{g_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n$ be chosen for $G, G^*$ and $\{h_i\}_{i=1}^m, \{\psi_i\}_{i=1}^m$ for $H, H^*$. Then define

$$(Pf)(t, s) = \sum_{i=1}^n \langle f, \phi_i \rangle g_i(s), \quad f \in L_1(T \times S),$$

$$(Qf)(t, *) = \sum_{i=1}^m \langle f^*, \psi_i \rangle h_i(t), \quad f \in L_1(T \times S).$$

These are (bounded, linear) projections of $L_1(T \times S)$ onto $U$ and $V$ respectively. It is easily verified that $PQ = QP$. By well-known results, $P + Q - PQ$ is a projection of $L_1(T \times S)$ onto $W$. The latter is therefore closed. Now given $w$ in $W$, we set $u = Pw - PQw$ and $v = Qw$, when $w = u + v$ is the required representation of $w$.

To prove the proximinality of $W$ in $L_1(T \times S)$, let $f$ be any element of $L_1(T \times S)$. Let $(w_n)$ be a minimising sequence for $f$; i.e. $\|f - w_n\| \to \text{dist}(f, W)$. We can assume without loss of generality that $\|w_n\| \leq 2\|f\|$ for all $n$. Then by Lemma E, we can write $w_n = u_n + v_n$ where $(u_n)$ and $(v_n)$ are bounded sequences in $U$ and $V$ respectively. Define $v_n^* = B_U u_n$ and $u_n^* = B_U v_n^*$.

$$\|f - u_n^* - v_n^*\| = \|f - v_n^* - A_U(f - v_n^*)\| \leq \|f - v_n^* - u_n\|$$

since $A_U(f - v_n^*)$ is the closest point in $U$ to $f - v_n^*$. Similarly,

$$\|f - u_n^* - v_n^*\| \leq \|f - u_n - v_n\| = \|f - u_n - A_V(f - u_n)\| \leq \|f - u_n - v_n\|.$$
Thus if $w^*_n = u^*_n + v^*_n$, $(w^*_n)$ is a minimising sequence for $f$. By Theorem D the set 
$\{w^*_n\}$ is weakly relatively compact. Furthermore, $W$ is closed by Lemma E and so 
$(w^*_n)$ has a weak cluster point $w$ in $W$. Since the norm is weakly lower semicontinuous, 
this point $w$ is a closest point to $f$ in $W$.

REFERENCES


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