FACTORISATION OF CHARACTERISTIC FUNCTIONS
ON NONCOMMUTATIVE GROUPS

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Abstract. A characteristic function, without idempotent factors, on a separable compact group is decomposed, modulo characters, as a product of indecomposable characteristic functions and an infinitely divisible characteristic function.

A continuous normalized positive definite function on a topological group $G$ will be called a characteristic function. Denote by $|\phi|^2$ the characteristic function defined by $|\phi|^2(g) = |\phi(g)|^2$ for all $g$ in $G$. The characteristic function identically 1 will be called degenerate. A continuous homomorphism of $G$ to $\mathbb{C}^*$, the group of complex numbers modulo 1, will be called a character. We are concerned with the factorisation of a characteristic function as a product of characteristic functions where we write $\phi = \phi_1\phi_2$ if $\phi(g) = \phi_1(g)\phi_2(g)$ for all $g$ in $G$. A characteristic function $\phi$ is called indecomposable if it cannot be expressed as a product of two other characteristic functions, idempotent if $\phi = \phi^2$ and infinitely divisible if for each $n \in \mathbb{N}$ one may write $\phi = \prod_{n=1}^{\infty} \phi^{(n)}$ for some characteristic function $\phi^{(n)}$, each $\phi^{(n)} = \phi^{(n)}$. Denote the set of factors of $\phi$ by $F_\phi$, the set of indecomposable factors of $\phi$ by $IF_\phi$ and the subgroup of $G$ generated by $\{g: \phi(g) \neq 0\}$ by $G_\phi$. Denote left Haar measure on a separable locally compact group by $dg$.

For the purposes of factorisation we shall consider two characteristic functions $\phi_1$ and $\phi_2$ to be equivalent if $\phi_1 = \phi_2\chi$ where $\chi$ is a character. When $G$ is commutative a characteristic function is the Fourier transform of a probability measure on the dual group $\hat{G}$ and equivalent characteristic functions are the Fourier transforms of shift-equivalent measures on $\hat{G}$ [4].

A. I. Khinchin [2] showed that the characteristic function of a probability distribution on $\mathbb{R}$ can be represented as $\phi_2\phi_3$ where $\phi_2$ is a denumerable product of indecomposable factors, $\phi_3$ has no indecomposable factors and is necessarily infinitely divisible. K. R. Parthasarathy, R. Ranga Rao and S. R. S. Varadhan [3] extended this result to a characteristic function on an arbitrary separable locally compact commutative group decomposing it as $\phi_1\phi_2\phi_3$ where $\phi_1$ is idempotent, $\phi_2$ and $\phi_3$ as above. When the group has no compact subgroups there is no proper idempotent factor.
The factorisation can be translated to the positive-definite matrices \([\alpha_{ij}] = [\phi(g, g^{-1})]\) for sequences \((g_i)\) in \(G\). The product of characteristic functions corresponds to coefficientwise multiplication of the matrices, and matrices \([\alpha_{ij}]\) and \([\beta_{ij}]\) correspond to equivalent characteristic functions if and only if \(\alpha_{ij} = \beta_{ij}c_ic_j\) for \(c_i, c_j \in \mathbb{C}^\ast\).

In §1 we consider the cancellation of idempotent factors from a characteristic function on a topological group and find conditions determining whether a characteristic function has idempotent factors or not. In §2 we prove Khinchin's factorisation theorem for a characteristic function, without idempotent factors, on a separable compact group. We have not been able to prove Khinchin's theorem for characteristic functions with idempotent factors, neither have we been able to construct a counterexample. In §3 we show why, in the commutative case, any characteristic function can be factorised as above.

1. Idempotent factors of a characteristic function on a topological group.

**Proposition 1.** Let \(G\) be a topological group. If \(\psi\) is an idempotent factor of a characteristic function \(\varphi\) of \(G\) then \(\psi = \chi_H\) where \(H\) is an open and closed subgroup of \(G\). The maximal idempotent factor, i.e. that with the minimal support and so the least degenerate, is \(\chi_G^\ast\). One factorises \(\varphi\) as \(\chi_G\varphi_0\) where \(\varphi_0\) is the restriction of \(\varphi\) to \(G_h^0\).

**Proof.** An idempotent is necessarily of the form \(x\varphi\) for a subset \(H\) of \(G\). Since the factors are required to be continuous it follows that \(H\) is open and closed and, since \(\psi(g_1) = 1, \psi(g_2) = 1\) implies \(\psi(g_1g_2) = 1\), \([1, 32.7]\), it follows that \(H\) must be an open (and closed) subgroup. For \(\chi_H\) to be a factor of \(\varphi\), necessarily \(\chi_H(g) \neq 0\) whenever \(\varphi(g) \neq 0\), so \(H \supseteq G_h^0\). By construction \(G_h^0\) is an open subgroup and so also closed. By \([1, 32.43]\), \(\varphi_0\) is also a characteristic function for \(G\).

**Corollary 1.** The characteristic function \(\varphi\) has nondegenerate idempotent factors if and only if \(G_h^0 \neq G\).

**Proposition 2.** Let \(\varphi\) be an infinitely divisible characteristic function on a group \(G\). It has a nondegenerate idempotent factor if and only if it has zeros.

**Proof.** The function \(\varphi_1 = \lim_n |\varphi|^{2^{-n}}\) is the idempotent factor, where \(|\varphi|(g) = |\phi(g)|\) for all \(g\) in \(G\). Indeed, \(\varphi\varphi\) is a characteristic function and, since \(\varphi\) is infinitely divisible, its repeated square roots will exist and be characteristic functions, Furthermore \(\varphi_1 = \chi_G^\ast\).

This generalises Lemma 4.2 of \([5]\), proved there for compact \(G\).

2. Characteristic functions, without idempotent factors, on a separable compact group.

**Lemma 1.** Let \(G\) be a separable compact group and \((\varphi_n)\) a sequence of characteristic functions such that \(\int |\varphi_n(g)|^2 \, dg \to 1\) as \(n \to \infty\). Then there exists a sequence \((\chi_n)\) of characters such that \(\varphi_n\chi_n\) converges uniformly to the degenerate characteristic function as \(n \to \infty\).
Proof. The proof is contained explicitly in the proof of [5, Lemma 4.1].

For a characteristic function \( \phi \), without idempotent factors, on a compact group we define the Khinchin functional \( N_\phi \) on \( F_\phi \), which measures 'departure' from the degenerate characteristic functional, by \( N_\phi(\psi) = -\int_G \log |\psi(g)| \, dg \). It is well defined and convergent since \( G \) is generated by a sequence of elements \( (g_i) \) such that \( \phi(g_i) \neq 0 \) for all \( i \), and since \( G \) is compact, \( N_\phi \) is bounded.

**Proposition 3.** Let \( \phi \) be a characteristic functional without idempotent factors on a separable compact group \( G \). If \( \psi_1, \psi_2, \psi \in F_\phi \)

(i) \( N_\phi(\psi_1\psi_2) = N_\phi(\psi_1) + N_\phi(\psi_2) \),

(ii) \( N_\phi(\psi) \geq \int_G (1 - |\psi(g)|) \, dg \geq 0 \),

(iii) \( N_\phi(\psi) = 0 \) if and only if \( \psi \) is equivalent to the degenerate characteristic function.

Proof. Properties (i) and (ii) are obvious. Property (iii) follows since \( N_\phi(\chi_G) = 0 \) and \( N_\phi(\chi) = 0 \) for any character \( \chi \); if \( N_\phi(\psi) = 0 \) then

\[
\int_G (1 - |\psi(g)|^2) \, dg \leq 2 \int_G (1 - |\psi(g)|) \, dg = 0
\]

by (ii), and, by Lemma 1, there exists a character \( \chi \) such that \( \psi \chi \) is degenerate.

**Lemma 2.** Let \( \phi \) be a characteristic function without idempotent factors on a separable compact group \( G \) and let \( \psi_i \) be a sequence of factors of \( \phi \) such that for all \( n \in \mathbb{N} \) the product \( \prod_{i=1}^n \psi_i \) is also a factor of \( \phi \). Then there exist characters \( \chi_i \) such that \( \prod_{i=1}^n \chi_i \psi_i \) converges to a characteristic function as \( n \to \infty \).

Proof. \( \sum N_\phi(\psi_i) \leq N_\phi(\phi) \) so \( \sum_{i=k}^\infty N_\phi(\psi_i) \) and \( N_\phi(\prod_{i=k}^\infty \psi_i) \) converge to zero as \( k \to \infty \). By Lemma 1 there exist \( (\chi_k) \) such that \( (\chi_{k-1} \prod_{i=k}^\infty \psi_i) \) converges to the degenerate characteristic function as \( k \to \infty \). Thus, absorbing each \( \chi_{k-1} \) in the preceding finite product, \( \prod_{i=1}^n \psi_i \chi_i \) converges to a characteristic function as \( n \to \infty \).

**Proposition 4.** Let \( \phi \) be a real-valued characteristic function on a compact group. Every sequence in \( F_\phi \) has a convergent subsequence.

Proof. The set \( F_\phi \) is equicontinuous since, for \( \psi \in F_\phi \),

\[
|\psi(g) - \psi(h)|^2 \leq 2(1 - \text{Re} \psi(g^{-1}h)) \leq 2(1 - \phi(g^{-1}h)).
\]

The proposition follows from the Arzela-Ascoli theorem.

Lemma 2 is the noncommutative version of [4, Theorem III.5.3], Proposition 4 is an analogue of Corollary III.5.2.

**Proposition 5.** Let \( G \) be a separable compact group. Any characteristic function without idempotent factors can be factorized, modulo a character, as a product of a denumerable number of indecomposable characteristic functions and a characteristic function with no indecomposable factors.

Proof. If \( \phi \) does not have any indecomposable factors the proposition holds. Suppose \( \phi \) has indecomposable factors. Write \( \text{Sup}(N_\phi(\psi) : \psi \in IF_\phi) = \delta(\phi) \). One can decompose \( \phi \) as \( \psi_1 \lambda_1 \) where \( N_\phi(\psi_1) \geq \frac{1}{2} \delta(\phi) \) and decompose the characteristic function \( \lambda_{n-1} \) as \( \psi_n \lambda_n \) where \( N_\phi(\psi_n) \geq \frac{1}{2} \delta(\lambda_{n-1}) \), for \( n = 2, 3, \ldots \). If \( \lambda_k \) has no
indecomposable factors for some \( k \) the process terminates and the proposition holds. When the process does not terminate there exist, by Lemma 2, characters \( \chi_i \) such that \( \prod \psi \chi_i \) converges. So \( N_q(\psi_n) \rightarrow 0 \) as \( n \rightarrow \infty \). So also \( \lambda_n \) will converge to a characteristic function \( \lambda \) as \( n \rightarrow \infty \). If \( \lambda \) has an indecomposable factor \( \psi \) then \( \psi \in F_{\lambda_n} \) for all \( n \) and so \( N_q(\psi) \leq \delta(\lambda_n) \) for all \( n \); as \( \delta(\lambda_n) \leq 2N_q(\psi_{n+1}) \rightarrow 0 \) as \( n \rightarrow \infty \), it follows from Proposition 3 that \( \psi \) is a character.

**Lemma 3.** Let \( \phi \) be a characteristic function, with no indecomposable factors and with no idempotent factors, on a separable compact group \( G \). There exists a sequence of decompositions \( (D_n) \) of \( \phi \) such that \( \nu = \inf_n \sup \{ 1 - |\phi(g)| : \phi \in D_n, g \in G \} = 0 \).

**Proof.** For any decomposition \( D \) of \( \phi \) let

\[
\nu_D = \sup \{ 1 - |\psi(g)| : \psi \in D, g \in G \}.
\]

For any characteristic function \( \tau \), if \( \psi \in F_\tau \) then \( 1 - |\psi(g)| \leq 1 - |\tau(g)| \) for all \( g \) in \( G \). One can arrange an array of decompositions

\[
(D_n: \phi = \phi_{n,1} \cdots \phi_{n,k_n})
\]

such that \( \nu_{D_n} \rightarrow \nu \) as \( n \rightarrow \infty \), \( 1 - |\phi_{n,j}(g)| \leq 1 - |\phi_{n,i}(g)| \) for all \( g \in G \), \( 1 < j \leq k_n \), and \( 1 - |\phi_{n,1}(g)| = \nu_{D_n} \) for some \( g \). Using Lemma 2, \( \phi \) can be decomposed as \( \phi_1 \phi_2 \), where \( \phi_2 = \lim_n \prod_{j=2}^{k_n} \chi_{n,j} \phi_{n,j} \), for an array \( (\chi_{n,j}) \) of characters of \( G \), and such that \( 1 - |\phi_{n}(g)| \leq \nu \) for some \( g \). Since \( \phi_1 \) and \( \phi_2 \) are again decomposable \( \nu \) must be 0.

An array of decompositions \( (D_n) \) such that \( \nu = 0 \) will be called uniformly infinitesimal.

**Corollary 2.** If a characteristic function \( \phi \) on a compact separable group has neither idempotent nor indecomposable factors then \( \{ g : \phi(g) \neq 0 \} = G_\phi \).

**Proof.** Since \( G_\phi = G_{\phi^2} \) it is sufficient to prove that if \( |\phi|^2(g_1) > 0 \) and \( |\phi|^2(g_2) > 0 \) then \( |\phi|^2(g_1 g_2) > 0 \). Choose a uniformly infinitesimal array of decompositions \( (\phi_{n,1} \cdots \phi_{n,k_n})_n \) of \( \phi \). For each of the decompositions \( |\phi|^2(g) = |\phi_{n,1}|^2(g) \cdots |\phi_{n,k_n}|^2(g) \). Thus \( |\phi|^2(g) > 0 \) if and only if, for any \( n \), \( |\phi_{n,j}|^2(g) > 0 \), \( 1 < j \leq k_n \), and so also if and only if \( \lim_n (n - |\phi_{n,j}|^2(g)) < \infty \) for \( j \in \mathbb{N} \). The corollary follows using [5, Lemma 3.6].

**Proposition 6.** A characteristic function \( \phi \), with neither idempotent nor indecomposable factors, on a compact separable group \( G \), is, modulo a character, infinitely divisible.

**Proof.** By Corollary 1, \( G = G_\phi \). We denote \( \phi(h^{-1} g)(\phi(h^{-1})\phi(g))^{-1} \) by \( K(g, h) \), adding suffixes if required. By Lemma 3 we can find a uniformly infinitesimal array \( (\phi_{n,1} \cdots \phi_{n,k_n})_n \) of decompositions of \( \phi \) such that, for large enough \( n \), \( 1 - |\phi_{n,j}(g)| \) is as small as we like. By [6, Lemma 3.5],

\[
|K_{n,j}(g, h)| \leq 2(1 - |\phi_{n,j}(h^{-1})|)^{1/2}(1 - |\phi_{n,j}(g)|)^{1/2}(\phi_{n,j}(h^{-1})\phi_{n,j}(g))^{-1}
\]

for \( n \in \mathbb{N} \), \( 1 < j \leq k_n \). So \( \lim_n \sup \{ 1 - K_{n,j}(g, h) \} = 0 \). Using the procedure of [6, Lemma 4.2] we can define \( L(g, h) = \log K(g, h) \) and prove it to be continuous and positive-definite on \( G \times G \). As in [6, Lemma 4.3], \( L(h, g^{-1}) \) is an additive 2-cocycle.
It is a coboundary since $H_2(G, \mathbb{R}) = \{0\}$ and the real and imaginary parts of $L$ can be considered separately. Hence $L(g, h) = \psi(h^{-1}g) - \psi(h^{-1}) - \psi(g)$ for some continuous conditionally positive-definite function $\psi$ on $G$. By [5, Theorem 4.1], $e^\psi$ is infinitely divisible. As in the proof of [6, Theorem 5.1], $e^\psi = \phi \chi$ for some character $\chi$ of $G$.

**Corollary 3.** On a separable compact group, if a characteristic function has no idempotent factors then it has indecomposable factors whenever it has zeros.

**Proof.** Suppose $\phi$ has zeros but no indecomposable factors. By Proposition 6 it is infinitely divisible so by Proposition 2 it cannot have zeros.

**Theorem.** Let $G$ be a separable compact group and $\phi$ a characteristic function on $G$ with no idempotent factors. Then $\phi$ can be decomposed, modulo a character, as a product of indecomposable characteristic functions and an infinitely divisible characteristic function.

**Proof.** The theorem follows from Propositions 5 and 6.

3. **Commutative groups.** The method in [3] for proving Lemma 2 for a locally compact separable commutative group is to use [4, Corollary III.5.2], the analogue of our Proposition 4, to prove the existence of characters $\chi_i$ such that products $\prod \psi_i \chi_i$ converge, and [4, Theorem III.5.2] to prove that all such convergent products are equivalent. Lemma 5 is [4, Theorem III.5.2] with a simpler proof than the original.

**Lemma 4.** Let $G$ be a complete separable metric commutative group. If $\phi$ and $\psi$ are characteristic functions on $G$ such that $\phi \psi$ is the degenerate characteristic function, then $\phi$ and $\psi$ are characters.

**Proof.** Denote the measure corresponding to a characteristic function $\xi$ by $\mu_\xi$. Since $\mu_\phi * \mu_\psi$ is the unit mass at the neutral element of $G$, so $\mu_\phi$ and $\mu_\psi$ must be point masses. Hence $\phi$ and $\psi$ are characters.

**Lemma 5.** Let $\phi$ and $\psi$ be characteristic functions on a complete separable metric commutative group. If $\phi \in F_\psi$ and $\psi \in F_\phi$, then $\phi$ is equivalent to $\psi$.

**Proof.** The lemma follows from Lemma 4. Indeed, if $\phi = \chi \psi$ and $\psi = \chi \phi$, then $\phi = \chi_1 \chi_2 \phi$, so $\chi_1 \chi_2$ is degenerate.

**Lemma 6.** Let $\phi$ be a characteristic function on a locally compact separable commutative group $G$. Any character on $G_\phi$ extends uniquely to a character of $G$.

**Proof.** Denote the annihilator of $G_\phi$ in $\hat{G}$ by $K$ and identify $\hat{G}_\phi = \hat{G}/K$ with a Borel section $B$ of $G$. As an element of $B$ is also an element of $G$, a character of $G_\phi$ uniquely determines a character of $G$.

Let $\phi$ be a characteristic function on a locally compact separable commutative group $G$. By Proposition 1, $\phi$ can be factorised as $\chi G_\phi \phi_0$. Propositions 5 and 6 hold for $\phi_0$ on $G_\phi$ [3]. By Lemma 6 the characters of $G_\phi$ occurring in the factorisation extend to characters of $G$. Thus $\phi$ can be factorised as $\phi_1 \phi_2 \phi_3$ as stated in the introduction.
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REFERENCES


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