GEOMETRIC REALIZATION OF $\pi_0 e(M)$

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Abstract. Let $M$ be a closed flat Riemannian manifold, $e(M)$ the group of self homotopy equivalences of $M$. Then there exists a subgroup $A(M)$ of $\text{Aff}(M)$ such that the natural homomorphism of $A(M)$ into $\pi_0 e(M)$ is a surjection with kernel a finite abelian group. Furthermore, this kernel can be identified with the structure group of the Calabi fibration.

This note is concerned with the realization problem of the group $\pi_0 e(M)$ of homotopy classes of self homotopy equivalences of $M$ as a group of affine diffeomorphisms for closed flat Riemannian manifolds $M$. We assume familiarity with [L-R1,2] and retain the notations and definitions given there. In [L-R1] it is shown that a finite abstract kernel $G \to \text{Out} \pi_1 M \cong \pi_0 e(M)$ can be realized as a group of affine diffeomorphisms of $M$ if and only if it admits an admissible group extension, i.e., if and only if there is a group extension $E$ of $\pi_1 M$ by $G$ realizing the abstract kernel, which is admissible (the centralizer of $\pi_1 M$ in $E$, $C_E(\pi_1 M)$, is torsion-free). See also [Z-Z].

When the center of $\pi_1 M$ is nontrivial, this condition is not automatically satisfied. In fact, for each $n$, there exist a closed flat manifold $M$ and a subgroup of $\text{Out} \pi_1 M$, isomorphic to $\mathbb{Z}_n$, which cannot be realized as any group of homeomorphisms of $M$. See [L-R2] for more details. Therefore, the next best thing for the realization problem would be finding an inflation of such a subgroup of $\pi_0 e(M)$ by a finite group, keeping the size of homotopy classes fixed, which can be realized. In fact, this can be done uniformly.

Main Theorem. Let $M$ be a closed flat Riemannian manifold. Then there exists a subgroup $A(M)$ of $\text{Aff}(M)$ such that the natural homomorphism of $A(M)$ into $\pi_0 e(M)$ is surjective and has a kernel isomorphic to the finite abelian group $H^1(M; \mathbb{Z})/\text{Center}(\pi_1 M)$.

The idea is to refine the Seifert fibered space construction designed by Corner and Raymond [C-R] in order to obtain certain criteria for a group to be embedded into the affine group. From now on, $M$ denotes a closed $n$-dimensional flat Riemannian manifold.
manifold, \( \pi_1 M = \pi \) and rank\( (\pi) = k \), where \( \pi \) is the center of \( \pi \). We consider \( \pi \) as a discrete uniform subgroup of \( E(n) \). Then \( \text{Inn}(\pi) = \pi/\pi(\pi) \) naturally becomes an \((n - k)\)-dimensional crystallographic group.

**Lemma 1.** There exists a faithful representation of \( \text{Aut}(\pi) \) into \( GL(k, \mathbb{Z}) \times A(n - k) \).

**Proof.** Since \( \pi(\pi) \) is characteristic in \( \pi \), any automorphism \( \theta \) of \( \pi \) induces an automorphism of \( \pi(\pi) \) and \( \text{Inn}(\pi) \), say \( \phi(\theta) \) and \( \tilde{\theta} \), respectively. A theorem of Bieberbach says that any automorphism of a crystallographic group is given by the conjugation by an affinity. This means \( \tilde{\theta} = \mu(h) \), conjugation by some \( h \in A(n - k) \). Furthermore, such an \( h \) is unique because \( \text{Inn}(\pi) \) has trivial center. Put \( \rho(\theta) = h \). Clearly \( \phi \) and \( \rho \) are homomorphisms.

It remains to show \( \phi \times \rho \) is injective. It is well known that the subgroup of automorphisms of \( \pi \) which induce the identity on both \( \pi(\pi) \) and \( \text{Inn}(\pi) \) is isomorphic to \( H^1(\text{Inn}(\pi); \pi(\pi)) \). See [We, 5.1.5], for example. However, \( \text{Inn}(\pi) \) acts trivially on \( \pi(\pi) \) and is centerless. This implies \( H^1(\text{Inn}(\pi); \pi(\pi)) = 0 \). Therefore, the injectivity of \( \phi \times \rho \) has been proved.

Now we describe the injective Seifert fibered space construction in a very special case. For the general construction, see P. Conner and F. Raymond’s articles, for example, [C-R]. See also the forthcoming paper [L-2] about results of restricting the coefficient space \( \text{MAPS}(W, R^k) \).

**Lemma 2.** Let \( Q \) be a group with a faithful representation \( \phi \times \rho: Q \to GL(k, \mathbb{Z}) \times A(n - k) \). Then there is an isomorphism of \( H^1(Q, T^k) \) onto the group of conjugacy classes (by \( R^k \)) of subgroups of \( A(k) \times A(n - k) \) which are extensions of \( Z^k \) by \( Q \), compatible with the representation \( \phi \times \rho \).

**Proof.** The action of \( Q \) on \( Z^k \) via \( \phi \) naturally extends to one on \( R^k \). The exact sequence of \( Q \)-modules \( 0 \to Z^k \to R^k \to T^k \to 0 \) induces a long exact sequence of cohomology \( \cdots \to H^1(Q, R^k) \to H^1(Q, T^k) \to H^2(Q, Z^k) \to \cdots \). Given \([m] \) \( \in H^1(Q, T^k) \), choose a 1-cocycle \( m: Q \to T^k \). Lift it to \( \tilde{m}: Q \to R^k \) taking care that \( \tilde{m}(1) = 0 \). Let \( E(\tilde{m}) = \{(\phi(\alpha), \tilde{m}(\alpha) + z), \rho(\alpha): \alpha \in Q, z \in Z^k \} \subset A(k) \times A(n - k) \subset A(n) \). We show that \( E(\tilde{m}) \) is a subgroup of \( A(n) \).

\[
(\phi(\alpha), \tilde{m}(\alpha) + z)(\phi(\beta), \tilde{m}(\beta) + z') = (\phi(\alpha\beta), \tilde{m}(\alpha) + z + \phi(\alpha)\tilde{m}(\beta) + \phi(\alpha)z') = (\phi(\alpha\beta), \tilde{m}(\alpha) + z + \phi(\alpha)z' + \delta\tilde{m}(\alpha, \beta)).
\]

Note that \( \delta\tilde{m}(\alpha, \beta) \in Z^k \) by construction so that \( z + \phi(\alpha)z' + \delta\tilde{m}(\alpha, \beta) \in Z^k \). In fact, \( E(\tilde{m}) \) is an extension of \( Z^k \) by \( Q \) representing the class \( \delta[m] \) \( \in H^2(Q, Z^k) \).

Suppose \( \tilde{m} = g + \delta\tilde{\lambda} \) for some \( g: Q \to Z^k \) and \( \tilde{\lambda} \in R^k \). Then \( E(\tilde{m}) = E(\delta\tilde{\lambda}) \) and \( \mu(\tilde{\lambda}) = \text{conjugation by } \tilde{\lambda} \) is an isomorphism of \( E(\tilde{\theta}) = Z^k \cdot Q \) to \( E(\delta\tilde{\lambda}) \). This proves that \( E(\tilde{m}) \) is unique up to conjugacy for any choice of \( m \in [m] \) and the lift \( \tilde{m} \). Conversely, if \( E(\tilde{m}) \) is conjugate to \( E(\tilde{\theta}) \) via \( \mu(\tilde{\lambda}) \), then \( E(\tilde{m}) = E(\delta\tilde{\lambda}) \), and hence \([m] = 0 \) in \( H^1(Q, T^k) \).
In order to show the surjectivity, let $E$ be a subgroup of $A(k) \times A(n-k)$ which is an extension of $\mathbb{Z}^k$ by $Q$. Furthermore, suppose the diagram

\[
\begin{array}{cccccc}
1 & \rightarrow & \mathbb{Z}^k & \rightarrow & E & \rightarrow & Q & \rightarrow & 1 \\
\downarrow & & \downarrow h & & \downarrow \phi & & \downarrow \rho & & \\
1 & \rightarrow & \mathbb{R}^k & \rightarrow & A(k) \times A(n-k) & \rightarrow & \text{GL}(k, \mathbb{Z}) \times A(n-k) & \rightarrow & 1
\end{array}
\]

commutes. We may assume that $E$ is $\mathbb{Z}^k \times Q$ with multiplication $(z, \alpha)(z', \beta) = (z + \phi(\alpha)z' + f(\alpha, \beta), \alpha \beta)$ for some 2-cocycle $f: Q \times Q \rightarrow \mathbb{Z}^k$. Define $\tilde{m}: Q \rightarrow \mathbb{R}^k$ by $\tilde{m}(\alpha) = \mathbb{R}^k$-component of $h(0, \alpha)$. Then clearly $\tilde{m}(\alpha, \beta) = \phi(\alpha)\tilde{m}(\beta) - \tilde{m}(\alpha \beta) + \tilde{m}(\alpha) = h(0, \alpha)h(0, \beta)h(0, \alpha \beta)^{-1} \in \mathbb{Z}^k$ so that $\tilde{m}: Q \rightarrow \mathbb{R}^k \rightarrow T^k$ is a coboundary. This shows $h(E) = E(\tilde{m})$ for $[\tilde{m}] \in H^1(Q, T^k)$. Therefore, the homomorphism of $H^1(Q, T^k)$ into the group of conjugacy classes of subgroups of $A(k) \times A(n-k)$ which satisfy the conditions is surjective.

**Remarks.** (1) The above construction does not require $\phi \times \rho(Q)$ to be discrete. (2) If $E$ and $E'$ are conjugate to each other by $\lambda \in \mathbb{R}^k$, one can deform one to the other by $\mu(t \lambda)$, $0 \leq t \leq 1$, in $A(k) \times A(n-k)$. (3) Not all extensions of $\mathbb{Z}^k$ by $Q$ can be imbedded in $A(k) \times A(n-k)$. This will be clear from the next

**Proposition.** An abstract kernel $(G, \pi, \varphi)$ can be realized as a group of affine diffeomorphisms of $M$ if and only if the kernel of $\varphi$ is finite and there exists an admissible extension $E$ of $\pi$ by $G$ realizing $\varphi$ so that $1 \rightarrow C_E(\pi) \rightarrow E \rightarrow E/C_E(\pi) \rightarrow 1$ has finite order in $H^2(E/C_E(\pi); C_E(\pi))$.

**Proof.** Let $1 \rightarrow \pi \rightarrow E \rightarrow G \rightarrow 1$ be an extension satisfying the conditions above. Since $0 \rightarrow \delta(\pi) \rightarrow C_E(\pi) \rightarrow \text{kernel}(\varphi) \rightarrow 1$ is exact, $C_E(\pi)$ is a torsion free central extension of $\delta(\pi)$ by the finite group, kernel($\varphi$), and hence is free abelian of rank $k$. (See the proof of [L-R1, Proposition 2].) In this proof we denote $C_E(\pi)$ by $\mathbb{Z}^k$ and $E/C_E(\pi)$ by $Q$. Since $\mu: E \rightarrow \text{Aut} \pi$ has kernel $C_E(\pi)$, we may consider $Q$ as a subgroup of $\text{Aut} \pi$. By Lemma 1, $Q$ has a representation $Q \rightarrow \text{GL}(k, \mathbb{Z}) \times A(n-k)$. We look at the long exact sequence in cohomology $\cdots \rightarrow H^1(Q, T^k) \rightarrow H^2(Q, \mathbb{Z}^k) \rightarrow H^2(Q, \mathbb{R}^k) \rightarrow \cdots$ induced by the exact sequence of $Q$-modules $0 \rightarrow \mathbb{Z}^k \rightarrow \mathbb{R}^k \rightarrow T^k \rightarrow 0$. Let $1 \rightarrow \mathbb{Z}^k \rightarrow E \rightarrow Q \rightarrow 1$ represents $[a] \in H^2(Q, \mathbb{Z}^k)$. Since $[a]$ has finite order, it is $0$ in $H^2(Q, \mathbb{R}^k)$. This implies that $[a] = \delta[m]$ for some $[m]$ in $H^1(Q, T^k)$. Then one can apply Lemma 2 to embed $E$ into $A(n)$. Let $\tilde{m}: E \rightarrow A(k) \times A(n-k)$ be such an embedding. Now, the trouble is that $\tilde{m}$ restricted to $\pi$ may not be the identity. However, note that the maximal abelian subgroup $\mathbb{Z}^n$ of $\pi$ maps into $\mathbb{R}^n$. Therefore, we have an isomorphism $\tilde{m}$ of a Bieberbach group $\pi$ into $A(n)$ so that $\tilde{m}(\mathbb{Z}^n) \subset \mathbb{R}^n$. It is proved in [L-R1, Theorem 6] that such an isomorphism is, in fact, a conjugation by an $h^{-1} \in A(n)$. Thus we have obtained a commutative diagram with exact rows,

\[
\begin{array}{cccccc}
1 & \rightarrow & \pi & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mu(h)\tilde{m}(E) & \rightarrow & \mu(h)\tilde{m}(E)/\pi & \rightarrow & 1
\end{array}
\]
Since \( \mu(h)m(E) \subset A(n) \), \( \mu(h)m(E)/\pi \subset \text{Aff}(M) \) which realizes the abstract kernel \( \varphi: G \to \text{Out} \pi \).

**Corollary [L-R1, Theorem 3; Z-Z, Satz 3.17].** A finite abstract kernel \( (G, \pi, \varphi) \) can be realized as a group of affine diffeomorphisms of \( M \) if and only if it admits an admissible extension.

**Proof.** We have to show that if \( 1 \to \pi \to E \to G \to 1 \) is an admissible extension, then \( 1 \to C_E(\pi) \to E \to E/C_E(\pi) \to 1 \) has a finite order in \( H^2(E/C_E(\pi); C_E(\pi)) \). This is a nontrivial fact, but can be shown by a cohomological argument which we leave to the reader.

**Proof of Main Theorem.** We have a natural homomorphism \( t: \pi \to \pi/[\pi, \pi] = H_1(\pi; \mathbb{Z}) \to H^1(\pi; \mathbb{Z}) \), where the last epimorphism is induced by the Universal Coefficient Theorem. Note that, under the homomorphism \( t \), \( \delta(\pi) \) injects into \( H^1(\pi; \mathbb{Z}) \). Since any automorphism of \( \pi \) induces an automorphism on \( H^1(\pi; \mathbb{Z}) \), \( \text{Aut} \pi \) acts on \( H^1(\pi; \mathbb{Z}) \). Form the semidirect product \( H^1(\pi; \mathbb{Z}) \circ \text{Aut} \pi \). We define a natural homomorphism \( \nu: \pi \to H^1(\pi; \mathbb{Z}) \circ \text{Aut} \pi \) by \( \nu(\sigma) = (t(\sigma), \mu(\sigma)) \). Noting that \( \text{Inn} \pi \) acts trivially on \( H^1(\pi; \mathbb{Z}) \), one can check easily that \( \nu \) is a homomorphism.

We claim that \( \nu \) is injective. Suppose \( \mu(\sigma) = 1 \). Then \( \sigma \in \delta(\pi) \). If \( t(\sigma) = 0 \), then \( \nu(p) \in [\pi, \pi] \) for some \( p > 0 \). Since \( \pi \) is torsion free and \( \delta(\pi) \cap [\pi, \pi] = 1 \), \( \nu(\sigma) = 1 \) implies \( \sigma = 1 \). Therefore, we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \delta(\pi) & H^1(\pi; \mathbb{Z}) & H^1(\pi; \mathbb{Z})/(\pi) & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \pi & H^1(\pi; \mathbb{Z}) \circ \text{Aut} \pi & A_1(M) & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \text{Inn} \pi & \text{Aut} \pi & \text{Out} \pi & 1 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Note that the abstract kernel \( \varphi: A_1(M) \to \text{Out} \pi \) is a finite inflation of \( \text{Out} \pi \) itself. In order words, kernel(\( \varphi \)) is a finite abelian group \( H^1(\pi; \mathbb{Z})/\delta(\pi) \). The middle row satisfies all the conditions in the proposition so that \( A_1(M) \) really sits inside \( \text{Aff}(M) \).

**Remark.** There is a certain relation between \( H^1(\pi; \mathbb{Z})/\delta(\pi) \) and the Calabi fibration. Calabi noted that \( M \) is covered by \( T^k \times N^{n-k} \), where \( T^k \) is a flat torus of dimension \( k = \text{rank} \delta(\pi) \) and \( N^{n-k} \) is a closed flat Riemannian manifold, with an abelian covering group \( A \). The group \( A \) acts on \( T^k \times N \) diagonally, as translations on \( T^k \) and as isometries on \( N \). See [W, Theorem 3.6.3] for more details. We can inflate \( A \)-action to \( A \times A \)-action on \( T^k \times N \) so that \( (\alpha, \beta)(x, y) = (\alpha x, \beta y) \) for \( (\alpha, \beta) \in A \times A \) and \( (x, y) \in T^k \times N \). Note that \( (A, T^k \times N) \) is naturally imbedded in \( (A \times A, T^k \times N) \) as the diagonal. Thus, \( (A \times A, T^k \times N) \) induces an action of \( A \times A/A \cong A \) on \( M \). It is not hard to verify that \( (H^1(M; \mathbb{Z})/\delta(\pi), M) \) is
EXACTLY THE SAME AS $(A \times A/A, M)$. THEREFORE THE GROUP $A_{i}(M)$ CONSTRUCTED IN THE THEOREM IS THE SMALLEST SUBGROUP OF $\text{Aff}(M)$ WHICH CONTAINS $A \times A/A$ AND MAPS ONTO $\text{Out } \pi$.

**COROLLARY.** GIVEN A FINITE SUBGROUP $G$ OF $\pi_{0}\epsilon(M)$, THERE ALWAYS EXISTS A GROUP $G^{*}$ TOGETHER WITH A SURJECTIVE HOMOMORPHISM $G^{*} \to G$ WITH A FINITE ABELIAN KERNEL SO THAT IT CAN BE REALIZED AS A GROUP OF AFFINE DIFFEOMORPHISMS OF $M$. FURTHERMORE, THE FINITE ABELIAN KERNELS ARE UNIFORMLY BOUNDED BY $H^{1}(M; \mathbb{Z})/\delta(\pi)$.


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