MAXIMAL INTERSECTING FAMILIES OF FINITE SETS 
AND \(n\)-UNIFORM HJELMSLEV PLANES

DAVID A. DRAKE\(^1\) AND SHARAD S. SANE

Abstract. The following theorem is proved. The collection of lines of an \(n\)-uniform projective Hjelmslev plane is maximal when considered as a collection of mutually intersecting sets of equal cardinality.

1. Introduction. A clique of \(k\)-sets is a collection of mutually intersecting sets of size \(k\). We write \(N(k)\) to denote the minimum cardinality of a maximal clique of \(k\)-sets. Apparently the exact value of \(N(k)\) is known only for very small values of \(k\). However, Erdős and Lovász [7] have obtained the asymptotic lower bound \(N(k) \geq \frac{8k}{3} - 3\); and Füredi [8, p. 283] writes that he can prove \(N(k) < k^{f(k)}\) where \(f(k) = ck^{7/12}\).

For particular values of \(k\), the preceding upper bound can be greatly sharpened. It is easily proved, for example, that

(1.1) a projective plane of order \(r\) is a maximal clique. Consequently

(1.2) \(N(r + 1) \leq r^2 + r + 1\) whenever \(r\) is the order of a projective plane.

In addition Füredi has proved the following two theorems (Proposition 1 and Theorem 1 in [8]). (Füredi informs us that (1.3) is joint work with L. Babai.)

(1.3) \(N(r^2 + r) \leq r^4 + r^3 + r^2\) whenever \(r\) is the order of a projective plane.

(1.4) \(N(2r) \leq 3r^2\) whenever \(r\) is the order of a projective plane.

In this paper we obtain the following common generalization of (1.2) and (1.3).

Theorem 1.1. If \(r\) is the order of a finite projective plane, then \(N(r^n + r^{n-1}) \leq r^{2n} + r^{2n-1} + r^{2n-2}\) for every positive integer \(n\).

Füredi proves (1.3) by constructing a 2-uniform projective Hjelmslev plane over an arbitrary finite projective plane and then observing that such Hjelmslev planes are maximal cliques. Henceforth we write PH-plane for projective Hjelmslev plane. The more difficult of the two steps in the Füredi program is the PH-plane construction, a construction which has been discovered independently by Füredi [8] and Craig [3] (see also Lüneburg [13]). Since the class of 1-uniform PH-planes is by definition just the class of finite projective planes, conclusions (1.2) and (1.3) both follow by observing that the line set of every \(n\)-uniform PH-plane with \(n = 1\) or \(2\) is a maximal clique. Similarly we shall obtain Theorem 1.1 as a corollary to the following result.
Theorem 1.2. The line set of every (finite) n-uniform projective Hjelmslev plane is a maximal clique.

The contribution of this paper is to prove Theorem 1.2. The other step, that of establishing the existence of n-uniform PH-planes over arbitrary projective planes, has already been completed: first by Artmann [1] and later by Drake [6] who used a different construction.

If k can be represented both as $r^m + r^{m-1}$ and as $s^p + s^{p-1}$ with $m < p$, one should apply Theorem 1.1 with $n = p$ to obtain the sharper bound. Unfortunately such double representations occur for prime powers $r$ and $s$ only when $m = 1$ and in the case $2^3 + 2^2 = 3^2 + 3 = 11 + 1$. In the latter case one obtains $N(12) \leq 133$ by using (1.2), $N(12) \leq 117$ by using the Füredi result (1.3), and $N(12) \leq 112$ by using Theorem 1.1 with $n = 3$. The real value of Theorem 1.1, of course, is that variation in $n$ allows one to obtain a bound for $N(k)$ for new values of $k$.

2. Prerequisites. We refer the reader to [5, pp. 192–197] for background material that includes the definitions of PH-planes and NAH-planes (near affine Hjelmslev planes). We repeat here some of the material from the cited pages, however, because the conclusions of this paper will interest a number of mathematicians without previous knowledge of Hjelmslev planes. We use the designation H-planes to refer collectively to NAH- and PH-planes.

To every H-plane $E$ is associated a canonical (incidence-structure) epimorphism $\phi: E \to E'$ where $E'$ is a projective plane if $E$ is a PH-plane and an affine plane if $E$ is an NAH-plane. Points $P$ and $Q$ (lines $g$ and $h$) are called neighbors, and one writes $P \sim Q$ ($g \sim h$), if and only if $P^\phi = Q^\phi$ ($g^\phi = h^\phi$). One writes $\sim$ for the negation of $\sim$. Intersecting lines $g$ and $h$ satisfy $g \sim h$ if and only if $|g \cap h| > 1$. We write $(P)$ to denote the set $\{Q: Q \sim P\}$ and $(g)$ to denote the set $\{h: h \sim g\}$. The following result was proved by Klingenberg [10, Satz 3.6]. (See also the remarks on page 260 of [12].)

Proposition 2.1. Let the incidence structure $A = A(H, h)$ be obtained from a PH-plane $H$ by removing a neighbor class ($h$) of lines as well as all points of $H$ which are incident with lines of ($h$). Then $A$ is an NAH-plane.

To each finite H-plane $E$ are associated three integers denoted by $r$, $s$ and $t$. For any flag $(P, g)$ the integer $t$ is the number of lines $h$ through $P$ which satisfy $h \sim g$ (as well as the number of points $Q$ on $g$ that satisfy $Q \sim P$); $|(P)| = |(g)| = t^2$; $s + t$ is the number of lines incident with $P$; and $r$ is the order of $E'$. Every line contains $s + t$ points if $E$ is a PH-plane, $s$ points if $E$ is an NAH-plane. The equality $s = rt$ holds for all H-planes. The preceding properties of $r$, $s$ and $t$ were first noted (for PH-planes only) by Kleinfeld [9]. Accordingly we shall designate this collection of properties the Kleinfeld Counting Lemma.

A nearly $1$-uniform PH-plane (NAH-plane) is a finite projective plane (finite affine plane). For $n > 1$ a finite H-plane $E$ (of either type) is called nearly $n$-uniform if, for every point $P$, (1) $E$ induces an incidence structure $A(P)$ on $(P)$ which is a nearly $(n - 1)$-uniform NAH-plane, (2) every line of $A(P)$ is induced by $d$ lines of $E$ for
some fixed integer \( d \). Proposition 1.10(11) of [5] asserts that \( d = r \). A nearly \( n \)-uniform H-plane is said to be \( n \)-uniform if every \( A(P) \) is an \((n - 1)\)-uniform NAH-plane with a "parallelism," but the reader will not need to understand this notion.

We now establish some conventions. All H-planes in this paper are assumed to be nearly \( n \)-uniform for some \( n \). The symbols \( E_n, H_n \) and \( A_n \) denote a nearly \( n \)-uniform H-, PH- and NAH-plane, respectively, with \( E', H' \) and \( A' \) as the respective underlying planes. In all cases the order of the underlying plane is assumed to be \( r \).

One writes \( P(\sim \ i)Q \) to mean that \( P \) and \( Q \) are joined by precisely \( r^i \) lines for \( 0 \leq i < n \) and \( P(\sim n)Q \) to mean that \( P = Q \). One writes \( P(\sim i)Q \) if \( P(\sim j)Q \) for some \( j \geq i \). The negation of \( P(\sim i)Q \) is denoted by \( P(\sim i)Q \). The following result is part of Proposition 1.10 of [5]; most of the proof, however, is given in the proof of Proposition 2.2 in [4] rather than in [5].

**Proposition 2.2.** Every nearly \( n \)-uniform H-plane \( E_n \) has the following properties.

1. \( s = r^n, t = r^{n-1} \).

2. If \( P \) and \( Q \) are distinct points of \( E_n \), then \( P(\sim i)Q \) for some nonnegative integer \( i < n \).

3. The dual of (2) holds for intersecting lines.

4. If \( P \) is in \( g \) and \( i \geq 1 \), then \( |\{Q \in g: Q(\sim i)P\}| = r^{n-i} \).

5. The dual of (4) holds.

One of the principal results of [4] (Proposition 4.6) asserts that the dual of a "strongly" \( n \)-uniform PH-plane is a strongly \( n \)-uniform PH-plane. In [14, Satz 1] Törner proves that every nearly \( n \)-uniform PH-plane is a strongly \( n \)-uniform PH-plane; Theorem 2.3 below is an immediate consequence. (An alternative proof is given in [11].)

**Theorem 2.3.** Every nearly \( n \)-uniform PH-plane is \( n \)-uniform, and the dual of an \( n \)-uniform PH-plane is an \( n \)-uniform PH-plane.

Two lines \( g \) and \( h \) of \( A_n \) are said to be quasiparallel (and one writes \( g \parallel h \)) if \( g^* \parallel h^* \) in \( A' \). Then \( | \) is an equivalence relation which partitions the lines of \( A_n \) into \( r + 1 \) quasiparallel classes; each such class is the disjoint union of \( r \) neighbor classes of lines, hence consists of \( rt^2 \) lines. As observed in [5, p. 202], the condition \( g \parallel h \) holds if and only if \( | g \cap h | \neq 1 \). This characterization of the quasiparallel relation makes it easy to prove the following lemma.

**Lemma 2.4.** Let \( g, h \) and \( P \) be lines and point of \( E_n \) such that \( g^* = g \cap (P) \) and \( h^* = h \cap (P) \) are not empty. Then \( g \sim h \) if and only if \( g^* \parallel h^* \) in \( A(P) \).

3. Preliminary results.

**Proposition 3.1.** Let \( \Lambda \) be a quasiparallel class of \( A_n \), \( S \subset \Lambda \), \( |S| < s = r^n \). Then there is a set \( C \) of points of \( A_n \) which has the following properties: (1) \( |C| = S \); (2) each pair of points of \( C \) is joined by a line of \( \Lambda \); (3) no point of \( C \) lies on any line of \( S \).
PROOF. For $n = 1$, $\Lambda$ is a parallel class, and $C$ may be taken to be the set of points of any line in $\Lambda \setminus S$. Assume $n > 1$, and let $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ be the $r$ line neighborhoods contained in $\Lambda$. If $S_j$ denotes $S \cap \Lambda_j$ for each $j$, then $|S_j| < s/r = t$ for some $i$. We intend to obtain $C$ from the set of points that are incident with lines of $\Lambda_i$. Let $h$ be a line in $\Lambda_i$; $P_1, P_2, \ldots, P_r$ be $r$ mutually nonneighbor points on $h$. For arbitrary fixed $j$, let $\Lambda^* = \{g': g' = g \cap (P_j) \text{ for some } g \in \Lambda_j\}$, $S^* = \{g': g' = g \cap (P_j) \text{ for some } g \in S_j\}$. By Lemma 2.4, $\Lambda^*$ is a quasiparallel class of lines in the nearly $(n-1)$-uniform NAH-plane $A(P_j)$; and $S'$ is a subset of fewer than $t = r^{n-1}$ lines of $\Lambda'$. By the induction assumption there is a set $C_j \subset (P_j)$ such that (1) $|C_j| = r^{n-1}$; (2) each pair of points of $C_j$ is joined by a line of $\Lambda'$; (3) no point of $C_j$ lies on any line of $S$. We take $C$ to be the union of the $C_j$.

PROPOSITION 3.2. Let $g$ be any line of $H_n$, $N \subset (g)$, $|N| < t$. Then there is a set $D$ of points of $H_n$ with the properties: (1) $|D| = s + t$; (2) each pair of points of $D$ is joined by a line of $(g)$; (3) no point of $D$ lies on any line of $N$.

PROOF. Let $P_0, P_1, \ldots, P_r$ be $r + 1$ mutually nonneighbor points on $g$. For fixed $j > 0$, apply Lemma 2.4 to see that the lines of $N$ induce a subset $N'$ of a quasiparallel class of lines in $A(P_j)$. Applying Proposition 3.1 (with $n - 1$ instead of $n$), we obtain a set $D_j$ of points of $(P_j)$ such that (1) $|D_j| = t$; (2) each pair of points of $D_j$ is joined by a line of $(g)$; (3) no point of $D_j$ lies on any line $N$. We now take $D$ to be the union of the $D_j$.

PROPOSITION 3.3. Let $S$ be a set of at most $s + t$ mutually intersecting lines of $A_n$ whose union contains every point of $A_n$. Then all lines of $S$ pass through a common point.

PROOF. The assertion is easily verified for $n = 1$, so assume $n > 1$. Let $g_1', g_2', \ldots, g_d'$ be the distinct images in $A'$ of the lines of $S$. Since the $g_i'$ intersect in $A'$, $d < r + 1$. Then the $g_i'$ pass through a common point $P'$, and hence the lines of $S$ all contain points from a common neighborhood $(P)$. The number of points of $A$, not in $(P)$, is $t^2(r^2 - 1) = s^2 - t^2$, and each line of $S$ contains $s - t$ points outside $(P)$. Then every point outside $(P)$ must lie on a single line of $S$, so every pair of lines of $S$ must intersect in $(P)$. Let $g$ be any line of $S$. Applying Proposition 2.2(5) with $i = n - 1$, one sees that there are $r - 1$ other lines $h$ which satisfy $h \cap (P) = g \cap (P)$. Take $Q$ to be any point of $h \setminus (P)$, and let $k$ be a line of $S$ which contains $Q$. Then $k$ and $g$ intersect in $g \cap (P) = h \cap (P)$. Then $k \cap h$ contains nonneighbor points, so $h = k$ is in $S$. It follows that the set $S^* = \{g \cap (P): g \in S\}$ has cardinality at most $(s + t)/r = r^{n-1} + r^{n-2}$. Applying the induction assumption to $A(P)$, we see that all lines of $S^*$ (and therefore all lines of $S$) meet in a common point.

4. Proofs of the main results. Thanks to Theorem 2.3, it is immaterial whether we prove Theorem 1.2 or its dual. Then let $S$ be a set of $s + t$ or fewer lines of $H_n$ whose union contains every point of $H_n$. To complete the proof of Theorem 1.2 it suffices to prove the existence of a point $P$ which lies on all lines of $S$. We intend to apply Proposition 3.3. To do so, we must remove a neighbor class $(h)$ of lines from $H_n$ to
obtain a nearly $n$-uniform NAH-plane $A_n$ (see Proposition 2.1). This must be done so that the intersections of lines of $S$ lie in $A_n$.

For any $g$ in $S$ let $N$ denote $S \cap (g)$. Assume $|N|<t$, and apply Proposition 3.2 to obtain a set $D$ of $s + t$ points. Conditions (2) and (3) of Proposition 3.2 guarantee that the points of $D$ lie on at least $s + t$ lines of $S \backslash (g)$. Since $g$ is in $S$, we have produced the contradiction $|S|>s + t$. Then $|S \cap (g)|$ must be at least $t$ for every $g$ in $S$, so $S$ contains lines from at most $(s + t)/t = r + 1$ distinct line neighborhoods of $H_n$. Consider the image $S^*$ of $S$ in $H'$, and apply the dual of (1.1): one sees that $S^*$ is the set of all $r + 1$ lines incident with some point $Q'$ of $H'$. Then $S$ contains exactly $t(r + 1) = s + t$ lines. Let $Q$ be a point of $H_n$ with $Q^* = Q'$. The number of flags $(R, g)$ with $g$ in $S$ and $R \sim Q$ is $(s + t)s = t^2(r^2 + r)$; i.e., is just the number of points $R$ of $H_n$ with $R \sim Q$. Then every point $R \sim Q$ lies on a unique line of $S$, so all intersections of pairs of lines of $S$ lie in $(Q)$. Let $h$ be any line having an empty intersection with $(Q)$. Applying Proposition 3.3 to $A_n = A(H_n, h)$ completes the proof of Theorem 1.2.

To prove Theorem 1.1, let $r$ be the order of a projective plane, $n$ be a positive integer. Then there exists an $n$-uniform PH-plane $H_n$ whose associated plane $H'$ is of order $r$: this assertion is Corollary 8 of [6]; it is also the main theorem of [1] if one uses the Bacon result [2] that finite PH-planes of level $n$ are $n$-uniform. Now Theorem 1.1 follows from Theorem 1.2 in view of Proposition 2.2(1) and the Kleinfeld Counting Lemma.

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