MAXIMAL INTERSECTING FAMILIES OF FINITE SETS
AND n-UNIFORM HJELMSLEV PLANES

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Abstract. The following theorem is proved. The collection of lines of an n-uniform projective Hjelmslev plane is maximal when considered as a collection of mutually intersecting sets of equal cardinality.

1. Introduction. A clique of k-sets is a collection of mutually intersecting sets of size k. We write N(k) to denote the minimum cardinality of a maximal clique of k-sets. Apparently the exact value of N(k) is known only for very small values of k. However, Erdös and Lovász [7] have obtained the asymptotic lower bound N(k) ≥ (8k/3) − 3; and Füredi [8, p. 283] writes that he can prove N(k) < k(2k) where f(k) = ck^{1/12}.

For particular values of k, the preceding upper bound can be greatly sharpened. It is easily proved, for example, that

1.1 a projective plane of order r is a maximal clique. Consequently
1.2 N(r + 1) ≤ r^2 + r + 1 whenever r is the order of a projective plane.

In addition Füredi has proved the following two theorems (Proposition 1 and Theorem 1 in [8]). (Füredi informs us that (1.3) is joint work with L. Babai.)
1.3 N(r^2 + r) ≤ r^4 + r^3 + r^2 whenever r is the order of a projective plane.
1.4 N(2r) ≤ 3r^2 whenever r is the order of a projective plane.

In this paper we obtain the following common generalization of (1.2) and (1.3).

Theorem 1.1. If r is the order of a finite projective plane, then N(r^n + r^{n-1}) ≤ r^{2n} + r^{2n-1} + r^{2n-2} for every positive integer n.

Füredi proves (1.3) by constructing a 2-uniform projective Hjelmslev plane over an arbitrary finite projective plane and then observing that such Hjelmslev planes are maximal cliques. Henceforth we write PH-plane for projective Hjelmslev plane. The more difficult of the two steps in the Füredi program is the PH-plane construction, a construction which has been discovered independently by Füredi [8] and Craig [3] (see also Lüneburg [13]). Since the class of 1-uniform PH-planes is by definition just the class of finite projective planes, conclusions (1.2) and (1.3) both follow by observing that the line set of every n-uniform PH-plane with n = 1 or 2 is a maximal clique. Similarly we shall obtain Theorem 1.1 as a corollary to the following result.
Theorem 1.2. The line set of every (finite) \( n \)-uniform projective Hjelmslev plane is a maximal clique.

The contribution of this paper is to prove Theorem 1.2. The other step, that of establishing the existence of \( n \)-uniform PH-planes over arbitrary projective planes, has already been completed: first by Artmann [1] and later by Drake [6] who used a different construction.

If \( k \) can be represented both as \( r^m + r^{m-1} \) and as \( s^p + s^{p-1} \) with \( m < p \), one should apply Theorem 1.1 with \( n = p \) to obtain the sharper bound. Unfortunately such double representations occur for prime powers \( r \) and \( s \) only when \( m = 1 \) and in the case \( 2^3 + 2^2 = 3^2 + 3 = 11 + 1 \). In the latter case one obtains \( N(12) \leq 133 \) by using (1.2), \( N(12) \leq 117 \) by using the Füredi result (1.3), and \( N(12) \leq 112 \) by using Theorem 1.1 with \( n = 3 \). The real value of Theorem 1.1, of course, is that variation in \( n \) allows one to obtain a bound for \( N(k) \) for new values of \( k \).

2. Prerequisites. We refer the reader to [5, pp. 192–197] for background material that includes the definitions of PH-planes and NAH-planes (near affine Hjelmslev planes). We repeat here some of the material from the cited pages, however, because the conclusions of this paper will interest a number of mathematicians without previous knowledge of Hjelmslev planes. We use the designation \( H \)-planes to refer collectively to NAH- and PH-planes.

To every \( H \)-plane \( E \) is associated a canonical (incidence-structure) epimorphism \( \phi: E \rightarrow E' \) where \( E' \) is a projective plane if \( E \) is a PH-plane and an affine plane if \( E \) is an NAH-plane. Points \( P \) and \( Q \) (lines \( g \) and \( h \)) are called neighbors, and one writes \( P \sim Q \) \((g \sim h)\), if and only if \( P^\phi = Q^\phi \) \((g^\phi = h^\phi)\). One writes \( \sim \) for the negation of \( \sim \). Intersecting lines \( g \) and \( h \) satisfy \( g \sim h \) if and only if \( |g \cap h| > 1 \). We write \( (P) \) to denote the set \( \{Q: Q \sim P\} \) and \( (g) \) to denote the set \( \{h: h \sim g\} \). The following result was proved by Klingenberg [10, Satz 3.6]. (See also the remarks on page 260 of [12].)

Proposition 2.1. Let the incidence structure \( A = A(H, h) \) be obtained from a PH-plane \( H \) by removing a neighbor class \((h)\) of lines as well as all points of \( H \) which are incident with lines of \((h)\). Then \( A \) is an NAH-plane.

To each finite \( H \)-plane \( E \) are associated three integers denoted by \( r, s \) and \( t \). For any flag \((P, g)\) the integer \( t \) is the number of lines \( h \) through \( P \) which satisfy \( h \sim g \) (as well as the number of points \( Q \) on \( g \) that satisfy \( Q \sim P \)); \( |(P)| = |(g)| = t^2 \); \( s + t \) is the number of lines incident with \( P \); and \( r \) is the order of \( E' \). Every line contains \( s + t \) points if \( E \) is a PH-plane, \( s \) points if \( E \) is an NAH-plane. The equality \( s = rt \) holds for all \( H \)-planes. The preceding properties of \( r, s \) and \( t \) were first noted (for PH-planes only) by Kleinfeld [9]. Accordingly we shall designate this collection of properties the Kleinfeld Counting Lemma.

A nearly \( 1 \)-uniform PH-plane (NAH-plane) is a finite projective plane (finite affine plane). For \( n > 1 \) a finite H-plane \( E \) (of either type) is called nearly \( n \)-uniform if, for every point \( P \), (1) \( E \) induces an incidence structure \( A(P) \) on \((P)\) which is a nearly \((n - 1)\)-uniform NAH-plane, (2) every line of \( A(P) \) is induced by \( d \) lines of \( E \) for
some fixed integer \( d \). Proposition 1.10(11) of [5] asserts that \( d = r \). A nearly \( n \)-uniform H-plane is said to be \( n \)-uniform if every \( A(P) \) is an \((n - 1)\)-uniform NAH-plane with a “parallelism,” but the reader will not need to understand this notion.

We now establish some conventions. All H-planes in this paper are assumed to be nearly \( n \)-uniform for some \( n \). The symbols \( E_n, H_n \) and \( A_n \) denote a nearly \( n \)-uniform H-, PH- and NAH-plane, respectively, with \( E', H' \) and \( A' \) as the respective underlying planes. In all cases the order of the underlying plane is assumed to be \( r \).

One writes \( P(\sim i)Q \) to mean that \( P \) and \( Q \) are joined by precisely \( r^i \) lines for \( 0 \leq i < n \) and \( P(\sim n)Q \) to mean that \( P = Q \). One writes \( P(\sim i)Q \) if \( P(\sim j)Q \) for some \( j \geq i \). The negation of \( P(\sim i)Q \) is denoted by \( P(\sim i)^cQ \). The following result is part of Proposition 1.10 of [5]; most of the proof, however, is given in the proof of Proposition 2.2 in [4] rather than in [5].

**Proposition 2.2.** Every nearly \( n \)-uniform H-plane \( E_n \) has the following properties.

1. \( s = r^n, t = r^{n-1} \).
2. If \( P \) and \( Q \) are distinct points of \( E_n \), then \( P(\sim i)Q \) for some nonnegative integer \( i < n \).
3. The dual of (2) holds for intersecting lines.
4. If \( P \) is in \( g \) and \( i \geq 1 \), then \( |\{Q \in g: Q(\sim i)P\}| = r^{n-i} \).
5. The dual of (4) holds.

One of the principal results of [4] (Proposition 4.6) asserts that the dual of a “strongly” \( n \)-uniform PH-plane is a strongly \( n \)-uniform PH-plane. In [14, Satz 1] Törner proves that every nearly \( n \)-uniform PH-plane is a strongly \( n \)-uniform PH-plane; Theorem 2.3 below is an immediate consequence. (An alternative proof is given in [11].)

**Theorem 2.3.** Every nearly \( n \)-uniform PH-plane is \( n \)-uniform, and the dual of an \( n \)-uniform PH-plane is an \( n \)-uniform PH-plane.

Two lines \( g \) and \( h \) of \( A_n \) are said to be quasiparallel (and one writes \( g \parallel h \)) if \( g^k \parallel h^k \) in \( A' \). Then \( \parallel \) is an equivalence relation which partitions the lines of \( A_n \) into \( r + 1 \) quasiparallel classes; each such class is the disjoint union of \( r \) neighbor classes of lines, hence consists of \( r^2 \) lines. As observed in [5, p. 202], the condition \( g \parallel h \) holds if and only if \( |g \cap h| \neq 1 \). This characterization of the quasiparallel relation makes it easy to prove the following lemma.

**Lemma 2.4.** Let \( g, h \) and \( P \) be lines and point of \( E_n \) such that \( g^c = g \cap (P) \) and \( h^c = h \cap (P) \) are not empty. Then \( g \sim h \) if and only if \( g^c \parallel h^c \) in \( A(P) \).

3. Preliminary results.

**Proposition 3.1.** Let \( \Lambda \) be a quasiparallel class of \( A_n \), \( S \subset \Lambda \), \( |S| < s = r^n \). Then there is a set \( C \) of points of \( A_n \) which has the following properties: (1) \( |C| = s \); (2) each pair of points of \( C \) is joined by a line of \( \Lambda \); (3) no point of \( C \) lies on any line of \( S \).
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Proof. For \( n = 1 \), \( \Lambda \) is a parallel class, and \( C \) may be taken to be the set of points of any line in \( \Lambda \setminus S \). Assume \( n > 1 \), and let \( \Lambda_1, \Lambda_2, \ldots, \Lambda_r \) be the \( r \) line neighborhoods contained in \( \Lambda \). If \( S_j \) denotes \( S \cap \Lambda_j \) for each \( j \), then \( |S_j| < s/r = t \) for some \( i \). We intend to obtain \( C \) from the set of points that are incident with lines of \( \Lambda_i \). Let \( h \) be a line in \( \Lambda_i \); \( P_1, P_2, \ldots, P_r \) be \( r \) mutually nonneighbor points on \( h \). For arbitrary fixed \( j \), let \( \Lambda' = \{g': g' = g \cap (P_j) \text{ for some } g \in \Lambda_j\} \), \( S' = \{g': g' = g \cap (P_j) \text{ for some } g \in S_j\} \). By Lemma 2.4, \( \Lambda' \) is a quasiparallel class of lines in the nearly \((n-1)\)-uniform NAH-plane \( A(P_j) \); and \( S' \) is a subset of fewer than \( t = r^{n-1} \) lines of \( \Lambda' \). By the induction assumption there is a set \( C_j \subseteq (P_j) \) such that (1) \( |C_j| = r^{n-1} \); (2) each pair of points of \( C_j \) is joined by a line of \( \Lambda \); (3) no point of \( C_j \) lies on any line of \( S \). We take \( C \) to be the union of the \( C_j \).

Proposition 3.2. Let \( g \) be any line of \( H_n \), \( N \subseteq (g) \), \( |N| < t \). Then there is a set \( D \) of points of \( H_n \) with the properties: (1) \( |D| = s + t \); (2) each pair of points of \( D \) is joined by a line of \( (g) \); (3) no point of \( D \) lies on any line of \( N \).

Proof. Let \( P_0, P_1, \ldots, P_r \) be \( r + 1 \) mutually nonneighbor points on \( g \). For fixed \( j > 0 \), apply Lemma 2.4 to see that the lines of \( N \) induce a subset \( N' \) of a quasiparallel class of lines in \( A(P_j) \). Applying Proposition 3.1 (with \( n - 1 \) instead of \( n \)), we obtain a set \( D_j \) of points of \( (P_j) \) such that (1) \( |D_j| = t \); (2) each pair of points of \( D_j \) is joined by a line of \( (g) \); (3) no point of \( D_j \) lies on any line of \( N \). We now take \( D \) to be the union of the \( D_j \).

Proposition 3.3. Let \( S \) be a set of at most \( s + t \) mutually intersecting lines of \( A_n \) whose union contains every point of \( A_n \). Then all lines of \( S \) pass through a common point.

Proof. The assertion is easily verified for \( n = 1 \), so assume \( n > 1 \). Let \( g_1', g_2', \ldots, g_d' \) be the distinct images in \( A' \) of the lines of \( S \). Since the \( g_j' \) intersect in \( A' \), \( d \leq r + 1 \). Then the \( g_j' \) pass through a common point \( P' \), and hence the lines of \( S \) all contain points from a common neighborhood \( (P) \). The number of points of \( A_n \) not in \( (P) \) is \( t^2(r^2 - 1) = s^2 - t^2 \), and each line of \( S \) contains \( s - t \) points outside \( (P) \). Then every point outside \( (P) \) must lie on a single line of \( S \), so every pair of lines of \( S \) must intersect in \( (P) \). Let \( g \) be any line of \( S \). Applying Proposition 2.2(5) with \( i = n - 1 \), one sees that there are \( r - 1 \) other lines \( h \) which satisfy \( h \cap (P) = g \cap (P) \). Take \( Q \) to be any point of \( h \setminus (P) \), and let \( k \) be a line of \( S \) which contains \( Q \). Then \( k \) and \( g \) intersect in \( g \cap (P) = h \cap (P) \). Then \( k \cap h \) contains nonneighbor points, so \( h = k \) is in \( S \). It follows that the set \( S^* = \{g \cap (P) : g \in S \} \) has cardinality at most \( (s + t)/r = r^{n-1} + r^{n-2} \). Applying the induction assumption to \( A(P) \), we see that all lines of \( S^* \) (and therefore all lines of \( S \)) meet in a common point.

4. Proofs of the main results. Thanks to Theorem 2.3, it is immaterial whether we prove Theorem 1.2 or its dual. Then let \( S \) be a set of \( s + t \) or fewer lines of \( H_n \) whose union contains every point of \( H_n \). To complete the proof of Theorem 1.2 it suffices to prove the existence of a point \( P \) which lies on all lines of \( S \). We intend to apply Proposition 3.3. To do so, we must remove a neighbor class \( (h) \) of lines from \( H_n \) to
obtain a nearly $n$-uniform NAH-plane $A_n$ (see Proposition 2.1). This must be done so that the intersections of lines of $S$ lie in $A_n$.

For any $g$ in $S$ let $N$ denote $S \cap (g)$. Assume $|N| < t$, and apply Proposition 3.2 to obtain a set $D$ of $s + t$ points. Conditions (2) and (3) of Proposition 3.2 guarantee that the points of $D$ lie on at least $s + t$ lines of $S \setminus \{g\}$. Since $g$ is in $S$, we have produced the contradiction $|S| > s + t$. Then $|S \cap (g)|$ must be at least $t$ for every $g$ in $S$, so $S$ contains lines from at most $(s + t)/t = r + 1$ distinct line neighborhoods of $H_n$. Consider the image $S^\circ$ of $S$ in $H'$, and apply the dual of (1.1): one sees that $S^\circ$ is the set of all $r + 1$ lines incident with some point $Q'$ of $H'$. Then $S$ contains exactly $t(r + 1) = s + t$ lines. Let $Q$ be a point of $H_n$ with $Q^\circ = Q'$. The number of flags $(R, g)$ with $g$ in $S$ and $R \sim Q$ is $(s + t)s = t^2(r^2 + r)$; i.e., is just the number of points $R$ of $H_n$ with $R \sim Q$. Then every point $R \sim Q$ lies on a unique line of $S$, so all intersections of pairs of lines of $S$ lie in $(Q)$. Let $h$ be any line having an empty intersection with $(Q)$. Applying Proposition 3.3 to $A_n = A(H_n, h)$ completes the proof of Theorem 1.2.

To prove Theorem 1.1, let $r$ be the order of a projective plane, $n$ be a positive integer. Then there exists an $n$-uniform PH-plane $H_n$ whose associated plane $H'$ is of order $r$: this assertion is Corollary 8 of [6]; it is also the main theorem of [1] if one uses the Bacon result [2] that finite PH-planes of level $n$ are $n$-uniform. Now Theorem 1.1 follows from Theorem 1.2 in view of Proposition 2.2(1) and the Kleinfeld Counting Lemma.

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REFERENCES


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