MAXIMAL INTERSECTING FAMILIES OF FINITE SETS
AND n-UNIFORM HJELMSLEV PLANES

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Abstract. The following theorem is proved. The collection of lines of an n-uniform projective Hjelmslev plane is maximal when considered as a collection of mutually intersecting sets of equal cardinality.

1. Introduction. A clique of k-sets is a collection of mutually intersecting sets of size k. We write N(k) to denote the minimum cardinality of a maximal clique of k-sets. Apparently the exact value of N(k) is known only for very small values of k. However, Erdős and Lovász [7] have obtained the asymptotic lower bound N(k) ≥ (8k/3) − 3; and Füredi [8, p. 283] writes that he can prove N(k) < k^{f(k)} where f(k) = ck^{7/12}.

For particular values of k, the preceding upper bound can be greatly sharpened. It is easily proved, for example, that

1.1 a projective plane of order r is a maximal clique. Consequently

1.2 N(r + 1) ≤ r^2 + r + 1 whenever r is the order of a projective plane.

In addition Füredi has proved the following two theorems (Proposition 1 and Theorem 1 in [8]). (Füredi informs us that (1.3) is joint work with L. Babai.)

1.3 N(r^2 + r) ≤ r^4 + r^3 + r^2 whenever r is the order of a projective plane.

1.4 N(2r) ≤ 3r^2 whenever r is the order of a projective plane.

In this paper we obtain the following common generalization of (1.2) and (1.3).

Theorem 1.1. If r is the order of a finite projective plane, then N(r^n + r^{n−1}) ≤ r^{2n} + r^{2n−1} + r^{2n−2} for every positive integer n.

Füredi proves (1.3) by constructing a 2-uniform projective Hjelmslev plane over an arbitrary finite projective plane and then observing that such Hjelmslev planes are maximal cliques. Henceforth we write PH-plane for projective Hjelmslev plane. The more difficult of the two steps in the Füredi program is the PH-plane construction, a construction which has been discovered independently by Füredi [8] and Craig [3] (see also Lüneburg [13]). Since the class of 1-uniform PH-planes is by definition just the class of finite projective planes, conclusions (1.2) and (1.3) both follow by observing that the line set of every n-uniform PH-plane with n = 1 or 2 is a maximal clique. Similarly we shall obtain Theorem 1.1 as a corollary to the following result.
Theorem 1.2. The line set of every (finite) n-uniform projective Hjelmslev plane is a maximal clique.

The contribution of this paper is to prove Theorem 1.2. The other step, that of establishing the existence of n-uniform PH-planes over arbitrary projective planes, has already been completed: first by Artmann [1] and later by Drake [6] who used a different construction.

If $k$ can be represented both as $r^m + r^{m-1}$ and as $s^p + s^{p-1}$ with $m < p$, one should apply Theorem 1.1 with $n = p$ to obtain the sharper bound. Unfortunately such double representations occur for prime powers $r$ and $s$ only when $m = 1$ and in the case $2^3 + 2 = 3^2 + 3 = 11 + 1$. In the latter case one obtains $N(12) \leq 133$ by using (1.2), $N(12) \leq 117$ by using the Füredi result (1.3), and $N(12) \leq 112$ by using Theorem 1.1 with $n = 3$. The real value of Theorem 1.1, of course, is that variation in $n$ allows one to obtain a bound for $N(k)$ for new values of $k$.

2. Prerequisites. We refer the reader to [5, pp. 192–197] for background material that includes the definitions of PH-planes and NAH-planes (near affine Hjelmslev planes). We repeat here some of the material from the cited pages, however, because the conclusions of this paper will interest a number of mathematicians without previous knowledge of Hjelmslev planes. We use the designation $H$-planes to refer collectively to NAH- and PH-planes.

To every $H$-plane $E$ is associated a canonical (incidence-structure) epimorphism $\phi: E \rightarrow E'$ where $E'$ is a projective plane if $E$ is a PH-plane and an affine plane if $E$ is an NAH-plane. Points $P$ and $Q$ (lines $g$ and $h$) are called neighbors, and one writes $P \sim Q$ ($g \sim h$), if and only if $P^* = Q^*$ ($g^* = h^*$). One writes $\sim$ for the negation of $\sim$. Intersecting lines $g$ and $h$ satisfy $g \sim h$ if and only if $|g \cap h| > 1$. We write $(P)$ to denote the set $\{Q: Q \sim P\}$ and $(g)$ to denote the set $\{h: h \sim g\}$. The following result was proved by Klingenberg [10, Satz 3.6]. (See also the remarks on page 260 of [12].)

Proposition 2.1. Let the incidence structure $A = A(H, h)$ be obtained from a PH-plane $H$ by removing a neighbor class $(h)$ of lines as well as all points of $H$ which are incident with lines of $(h)$. Then $A$ is an NAH-plane.

To each finite H-plane $E$ are associated three integers denoted by $r$, $s$ and $t$. For any flag $(P, g)$ the integer $t$ is the number of lines $h$ through $P$ which satisfy $h \sim g$ (as well as the number of points $Q$ on $g$ that satisfy $Q \sim P$); $|(P)| = |(g)| = t^2$; $s + t$ is the number of lines incident with $P$; and $r$ is the order of $E'$. Every line contains $s + t$ points if $E$ is a PH-plane, $s$ points if $E$ is an NAH-plane. The equality $s = rt$ holds for all $H$-planes. The preceding properties of $r$, $s$ and $t$ were first noted (for PH-planes only) by Kleinfeld [9]. Accordingly we shall designate this collection of properties the Kleinfeld Counting Lemma.

A nearly 1-uniform PH-plane (NAH-plane) is a finite projective plane (finite affine plane). For $n > 1$ a finite H-plane $E$ (of either type) is called nearly n-uniform if, for every point $P$, (1) $E$ induces an incidence structure $A(P)$ on $(P)$ which is a nearly $(n - 1)$-uniform NAH-plane, (2) every line of $A(P)$ is induced by $d$ lines of $E$ for
some fixed integer $d$. Proposition 1.10(11) of [5] asserts that $d = r$. A nearly $n$-uniform H-plane is said to be $n$-uniform if every $A(P)$ is an $(n - 1)$-uniform NAH-plane with a "parallelism," but the reader will not need to understand this notion.

We now establish some conventions. All H-planes in this paper are assumed to be nearly $n$-uniform for some $n$. The symbols $E_n$, $H_n$ and $A_n$ denote a nearly $n$-uniform H-, PH- and NAH-plane, respectively, with $E'$, $H'$ and $A'$ as the respective underlying planes. In all cases the order of the underlying plane is assumed to be $r$.

One writes $P(\sim i)Q$ to mean that $P$ and $Q$ are joined by precisely $r^i$ lines for $0 \leq i < n$ and $P(\sim n)Q$ to mean that $P = Q$. One writes $P(\sim i)Q$ if $P(\sim j)Q$ for some $j \geq i$. The negation of $P(\sim i)Q$ is denoted by $P(\sim i)Q$. The following result is part of Proposition 1.10 of [5]; most of the proof, however, is given in the proof of Proposition 2.2 in [4] rather than in [5].

**Proposition 2.2.** Every nearly $n$-uniform H-plane $E_n$ has the following properties. 
(1) $s = r^n$, $t = r^{n-1}$.
(2) If $P$ and $Q$ are distinct points of $E_n$, then $P(\sim i)Q$ for some nonnegative integer $i < n$.
(3) The dual of (2) holds for intersecting lines.
(4) If $P$ is in $g$ and $i \geq 1$, then $\{Q \in g: Q(\sim i)P\} = r^{n-i}$.
(5) The dual of (4) holds.

One of the principal results of [4] (Proposition 4.6) asserts that the dual of a "strongly" $n$-uniform PH-plane is a strongly $n$-uniform PH-plane. In [14, Satz 1] Törner proves that every nearly $n$-uniform PH-plane is a strongly $n$-uniform PH-plane; Theorem 2.3 below is an immediate consequence. (An alternative proof is given in [11].)

**Theorem 2.3.** Every nearly $n$-uniform PH-plane is $n$-uniform, and the dual of an $n$-uniform PH-plane is an $n$-uniform PH-plane.

Two lines $g$ and $h$ of $A_n$ are said to be quasiparallel (and one writes $g \parallel h$) if $g^* \parallel h^*$ in $A'$. Then $\parallel$ is an equivalence relation which partitions the lines of $A_n$ into $r + 1$ quasiparallel classes; each such class is the disjoint union of $r$ neighbor classes of lines, hence consists of $rt^2$ lines. As observed in [5, p. 202], the condition $g \parallel h$ holds if and only if $|g \cap h| \neq 1$. This characterization of the quasiparallel relation makes it easy to prove the following lemma.

**Lemma 2.4.** Let $g$, $h$ and $P$ be lines and point of $E_n$ such that $g' = g \cap (P)$ and $h' = h \cap (P)$ are not empty. Then $g \parallel h$ if and only if $g' \parallel h'$ in $A(P)$.

3. Preliminary results.

**Proposition 3.1.** Let $\Lambda$ be a quasiparallel class of $A_n$, $S \subset \Lambda$, $|S| < s = r^n$. Then there is a set $C$ of points of $A_n$ which has the following properties: (1) $|C| = s$; (2) each pair of points of $C$ is joined by a line of $\Lambda$; (3) no point of $C$ lies on any line of $S$.
Proof. For \( n = 1 \), \( \Lambda \) is a parallel class, and \( C \) may be taken to be the set of points of any line in \( \Lambda \) \( \setminus \) \( S \). Assume \( n > 1 \), and let \( \Lambda_1, \Lambda_2, \ldots, \Lambda_r \) be the \( r \) line neighborhoods contained in \( \Lambda \). If \( S_j \) denotes \( S \cap \Lambda_j \) for each \( j \), then \( |S_j| < s/r = t \) for some \( i \). We intend to obtain \( C \) from the set of points that are incident with lines of \( \Lambda_i \). Let \( h \) be a line in \( \Lambda_i \); \( P_1, P_2, \ldots, P_r \) be \( r \) mutually nonneighbor points on \( h \). For arbitrary fixed \( j \), let \( \Lambda_j' = \{ g' : g' = g \cap (P_j) \text{ for some } g \in \Lambda_j \} \), \( S_j' = \{ g' : g' = g \cap (P_j) \text{ for some } g \in S_j \} \). By Lemma 2.4, \( \Lambda_j' \) is a quasiparallel class of lines in the nearly \((n - 1)\)-uniform NAH-plane \( A(P_j) \): and \( S_j' \) is a subset of fewer than \( t = r^{n-1} \) lines of \( \Lambda_j' \). By the induction assumption there is a set \( C_j \subseteq (P_j) \) such that (1) \( |C_j| = r^{n-1} \); (2) each pair of points of \( C_j \) is joined by a line of \( \Lambda \); (3) no point of \( C_j \) lies on any line of \( S \). We take \( C \) to be the union of the \( C_j \).

**Proposition 3.2.** Let \( g \) be any line of \( H_n \), \( N \subseteq (g) \), \( |N| < t \). Then there is a set \( D \) of points of \( H_n \) with the properties: (1) \( |D| = s + t \); (2) each pair of points of \( D \) is joined by a line of \((g)\); (3) no point of \( D \) lies on any line of \( N \).

Proof. Let \( P_0, P_1, \ldots, P_r \) be \( r + 1 \) mutually nonneighbor points on \( g \). For fixed \( j \geq 0 \), apply Lemma 2.4 to see that the lines of \( N \) induce a subset \( N_j \) of a quasiparallel class of lines in \( A(P_j) \). Applying Proposition 3.1 (with \( n - 1 \) instead of \( n \)), we obtain a set \( D_j \) of points of \((P_j)\) such that (1) \( |D_j| = t \); (2) each pair of points of \( D_j \) is joined by a line of \((g)\); (3) no point of \( D_j \) lies on any line of \( N \). We now take \( D \) to be the union of the \( D_j \).

**Proposition 3.3.** Let \( S \) be a set of at most \( s + t \) mutually intersecting lines of \( A_n \) whose union contains every point of \( A_n \). Then all lines of \( S \) pass through a common point.

Proof. The assertion is easily verified for \( n = 1 \), so assume \( n > 1 \). Let \( g_1', g_2', \ldots, g_d' \) be the distinct images in \( A' \) of the lines of \( S \). Since the \( g_i' \) intersect in \( A' \), \( d \leq r + 1 \). Then the \( g_i' \) pass through a common point \( P' \), and hence the lines of \( S \) all contain points from a common neighborhood \((P)\). The number of points of \( A_1 \) not in \((P)\) is \( r^2(r^2 - 1) = s^2 - t^2 \), and each line of \( S \) contains \( s - t \) points outside \((P)\). Then every point outside \((P)\) must lie on a single line of \( S \), so every pair of lines of \( S \) must intersect in \((P)\). Let \( g \) be any line of \( S \). Applying Proposition 2.2(5) with \( i = n - 1 \), one sees that there are \( r - 1 \) other lines \( h \) which satisfy \( h \cap (P) = g \cap (P) \). Take \( Q \) to be any point of \( h \setminus (P) \), and let \( k \) be a line of \( S \) which contains \( Q \). Then \( k \) and \( g \) intersect in \( g \cap (P) = h \cap (P) \). Then \( k \cap (P) \) contains nonneighbor points, so \( h = k \) is in \( S \). It follows that the set \( S^* = \{ g \cap (P) : g \in S \} \) has cardinality at most \((s + t)/r = r^{n-1} + r^{n-2} \). Applying the induction assumption to \( A(P) \), we see that all lines of \( S^* \) (and therefore all lines of \( S \)) meet in a common point.

4. Proofs of the main results. Thanks to Theorem 2.3, it is immaterial whether we prove Theorem 1.2 or its dual. Then let \( S \) be a set of \( s + t \) or fewer lines of \( H_n \) whose union contains every point of \( H_n \). To complete the proof of Theorem 1.2 it suffices to prove the existence of a point \( P \) which lies on all lines of \( S \). We intend to apply Proposition 3.3. To do so, we must remove a neighbor class \((h)\) of lines from \( H_n \) to...
obtain a nearly n-uniform NAH-plane $A_n$ (see Proposition 2.1). This must be done so that the intersections of lines of $S$ lie in $A_n$.

For any $g$ in $S$ let $N$ denote $S \cap (g)$. Assume $|N| < t$, and apply Proposition 3.2 to obtain a set $D$ of $s + t$ points. Conditions (2) and (3) of Proposition 3.2 guarantee that the points of $D$ lie on at least $s + t$ lines of $S \setminus (g)$. Since $g$ is in $S$, we have produced the contradiction $|S| > s + t$. Then $|S \cap (g)|$ must be at least $t$ for every $g$ in $S$, so $S$ contains lines from at most $(s + t)/t = r + 1$ distinct line neighborhoods of $H_n$. Consider the image $S^g$ of $S$ in $H'$, and apply the dual of (1.1): one sees that $S^g$ is the set of all $r + 1$ lines incident with some point $Q'$ of $H'$. Then $S$ contains exactly $t(r + 1) = s + t$ lines. Let $Q$ be a point of $H_n$ with $Q'^g = Q'$. The number of flags $(R, g)$ with $g$ in $S$ and $R \sim Q$ is $(s + t)s = t^2(r^2 + r)$; i.e., is just the number of points $R$ of $H_n$ with $R \sim Q$. Then every point $R \sim Q$ lies on a unique line of $S$, so all intersections of pairs of lines of $S$ lie in $(Q)$. Let $h$ be any line having an empty intersection with $(Q)$. Applying Proposition 3.3 to $A_n = A(H_n, h)$ completes the proof of Theorem 1.2.

To prove Theorem 1.1, let $r$ be the order of a projective plane, $n$ be a positive integer. Then there exists an $n$-uniform PH-plane $H_n$ whose associated plane $H'$ is of order $r$: this assertion is Corollary 8 of [6]; it is also the main theorem of [1] if one uses the Bacon result [2] that finite PH-planes of level $n$ are $n$-uniform. Now Theorem 1.1 follows from Theorem 1.2 in view of Proposition 2.2(1) and the Kleinfeld Counting Lemma.

ACKNOWLEDGMENT. During the writing of this paper, the second author held a visiting position in the Department of Mathematics at the University of Florida.

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