DIEUDONNÉ-SCHWARTZ THEOREM ON BOUNDED SETS IN INDUCTIVE LIMITS. II

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Abstract. The Dieudonné-Schwartz Theorem [1, Chapter 2, §12] has been stated for strict inductive limits. In [3] it has been extended to inductive limits. Here the result of [3] is generalized. Also, the case when each set bounded in \( \text{ind lim } E_n \) is contained, but not necessarily bounded, in some \( E_n \) is considered.

Let \( E_1 \subset E_2 \subset \cdots \) be a sequence of locally convex spaces and \( E = \text{ind lim } E_n \) their inductive limit (with respect to the identity maps \( \text{id}: E_n \to E_{n+1} \)). The Dieudonné-Schwartz theorem states that a set \( B \subset E \) is bounded if and only if it is contained and bounded in some \( E_n \), provided that

- \( \text{(H-1)} \) each \( E_n \) is closed in \( E_{n+1} \), and
- \( \text{(H-2)} \) the topology of each \( E_n \) equals the topology induced in \( E_n \) by \( E_{n+1} \). It is convenient to introduce some further hypotheses:

- \( \text{(H-3)} \) each \( E_n \) is closed in \( E \),
- \( \text{(H-4)} \) each convex and closed set in \( E_n \) is closed in \( E_{n+1} \),
- \( \text{(H-7)} \) for any \( n \in \mathbb{N} \) there is \( p \in \mathbb{N} \) such that \( E_n \subset E_{n+p} \), where \( E_n^E \) is the closure of \( E_n \) in \( E \),
- \( \text{(H-8)} \) for any closed hyperplane \( F \) in \( E_n \), \( (E_n \setminus F) \cap E_{n+1}^E = \emptyset \),
- \( \text{(DS)} \) each set \( B \) bounded in \( E \) is contained in some \( E_n \), and
- \( \text{(DST)} \) each set \( B \) bounded in \( E \) is contained and bounded in some \( E_n \).

The following implications: \( \text{H-1 & 2} \Rightarrow \text{H-3} \), \( \text{H-3} \Rightarrow \text{DS} \), \( \text{H-4} \Rightarrow \text{DST} \), and \( \text{H-4} \Rightarrow \text{H-3} \), are known, see [1, Chapter 2, §12; 2 and 3].

Theorem 1. \( \text{H-7} \Rightarrow \text{DS} \). If \( E \) is metrizable, the implication can be reversed.

Proof. Assume \( \text{H-7} \) and existence of a set \( B \) bounded in \( E \) which is not contained in any \( E_n \). Choose a sequence \( 1 = n_1 \leq n_2 \leq n_3 \leq \cdots \) such that \( E_{n_k}^E \subset E_{n_{k+1}} \) and \( b_k \in B \setminus E_{n_k}, k \in \mathbb{N} \).

Since \( b_1 \neq 0 \), there exists convex 0-nbhd \( G_1 \) in \( E \) such that \( b_1 \notin G_1 + G_1 \). Put \( V_1 = G_1 \cap E_{n_1} \) and \( W_1 = V_1^E \). Then \( W_1 \subset (G_1 + G_1) \cap E_{n_2} \) and \( b_1 \notin W_1, \frac{1}{2} b_2 \notin W_1 \). Hence there exists convex 0-nbhd \( G_2 \) in \( E \) such that \( b_1, \frac{1}{2} b_2 \notin W_1 + G_2 + G_2 \). Put \( V_2 = G_2 \cap E_{n_2} \) and \( W_2 = V_2 + V_2^E \). Again \( W_2 \subset (W_1 + G_2 + G_2) \cap E_{n_3} \), and \( b_1, \frac{1}{2} b_2, \frac{1}{3} b_3 \notin W_2 \), etc. When the sequence \( \{W_k\} \) is constructed, then \( W = \bigcup \{W_k; k \in \mathbb{N}\} \) is a 0-nbhd in \( E \) which does not absorb \( B \).
Let \( \{G_p\} \) be a nested base for the topology of \( E \). Assume \( E^E_1 \) is not contained in any \( E_p \). Take \( x_p \in E^E_1 \setminus E_p \) and \( a_p > 0 \) such that \( a_p x_p \in G_p \), \( p \in N \). Then \( B = \bigcup \{a_p x_p, p \in N\} \) is bounded in \( E \) and not contained in any \( E_p \).

**Lemma 1.** \( H-8 \Leftrightarrow \) each \( g \in E_n' \) has a continuous extension to \( E_{n+1} \).

**Proof.** Assume \( H-8 \) and take \( g \in E_n' \), \( f \neq 0 \). Choose \( x_0 \in E_n \), \( f(x_0) \neq 0 \) and put \( F = F^{-1}(0) \). Since, by \( H-8 \), \( x_0 \notin E_{n+1}' \) there exists \( g \in E_{n+1}' \) such that \( g(x_0) = f(x_0) \) and \( g(x) = 0 \) for \( x \in F_{E_{n+1}} \), that is \( g^{-1}(0) \supset F \) and \( g \) is the sought extension of \( g \).

Let \( F \) be a closed hyperplane in \( E_n \). Take \( f \in E_n' \) such that \( f^{-1}(0) = F \). If \( f \) has an extension \( g \) to \( E_{n+1} \) then for \( x \in E_n \setminus G \), \( g(x) = f(x) \neq 0 \), and \( x \notin g^{-1}(0) = F_{E_{n+1}} \).

**Lemma 2.** \( DS \& H-8 \Rightarrow \) each set \( B \subseteq E_n \) which is bounded in \( E \) is bounded in \( E_n \).

**Proof.** Assume \( B \subseteq E_n \), bounded in \( E \), but not bounded in \( E_n \). Then \( B \) is not weakly bounded in \( E_n \) and there is \( f_0 \in E_n' \) (real dual) which is not bounded on \( B \). For each \( k \in N \), take \( b_k \in B, f_0(b_k) > k \). By induction, choose \( f_p \in E_{n+p}' \) so that \( f_p \) is an extension of \( f_{p-1} \), \( p \in N \). Then \( \bigcup \{f_p^{-1}(-\infty, 1); p \in N\} \) is a 0-nbd in \( E \) which does not absorb \( B \).

From Theorem 1 and Lemmas 1 and 2 it follows that:

**Theorem 2.** \( H-7 \& 8 \Rightarrow DS \& H-8 \Rightarrow DST \).

**Proposition.** \( H-4 \Leftrightarrow H-3 \& 8 \Leftrightarrow H-1 \& 8 \).

**Proof.** Evidently the if implications hold. To complete the cycle, assume \( H-1 \& 8 \). Take a set \( A \) closed and convex in \( E_n \). Without loss of generality, we may assume \( 0 \in A \). Denote by \( g_f \) a continuous extension of \( f \in E_n' \) to \( E_{n+1} \). There exists \( M \subseteq E_n' \) such that \( A = \bigcap \{f^{-1}(-\infty, 1); f \in M\} = \bigcap \{g_f^{-1}(-\infty, 1); f \in M\} \cap E_n \supset \bar{A}^E_{n+1} \), since \( E_n \) is closed in \( E_{n+1} \).

We have a diagram:

\[
\begin{array}{ccc}
3 \& 8 & \Rightarrow & 7 \& 8 & \Rightarrow & DS \& 8 \\
3 & \Downarrow & & \Downarrow & & \Downarrow & \\
1 \& 2 & \Rightarrow & 4 & \Rightarrow & 7 \& 8 & \Rightarrow & DST \\
& \Downarrow & & \Downarrow & \Downarrow & \Downarrow & \\
3 & \Rightarrow & 7 & \Rightarrow & DS
\end{array}
\]

The following examples will show that \( H-7 \& 8 \) do not imply \( H-4 \) and \( DST \& H-8 \) do not imply \( H-7 \).

**Example 1.** Take a Banach space \( X \) and its proper subspace \( Y \) (with the inherited topology). Put \( E_{2n-1} = X^n \times \{0\}^N \), \( E_{2n} = X^n \times Y \times \{0\}^N \), \( n \in N \), all with the product topology. Then \( E = \bigcup \{E_n, n \in N\} \subset X^N \) has the topology inherited from \( X^N \), as well as all \( E_n \). Hence \( H-8 \) holds. Further \( E_{2n+1}^E = E_{2n+1} = E_{2n+1} \) and \( H-7 \) holds. On the other hand, \( H-3 \& 4 \) do not hold, since \( E_{2n+1} = E_{2n+1}' = E_{2n+1} \).

**Example 2.** Let \( \mathcal{D}[-n, n] = \{f \in C^\infty(R); \supp f \subseteq [-n, n]\} \) and \( \mathcal{D} = \text{indlim} \mathcal{D}[-n, n] \). For this inductive limit \( DST \) holds by Dieudonné-Schwartz Theorem. Take \( \varphi \in \mathcal{D} \), \( \supp \varphi = [-1, 1] \), \( A = \{\varphi((p + 1)x/pq); p, q \in N\} \), and put \( E_n = \text{sp}(A \cup \mathcal{D}[-n, n]), n \in N \), where \( \text{sp} \) stands for the span. We equip each \( E_n \).
with the topology inherited from $\mathcal{D}$ and $H-8$ holds. Since $\mathcal{D}[-n, n] \subset E_n$, DST holds for the ind lim $E_n$. On the other hand the closure of $E_n$ in $E$ contains functions $\varphi(\frac{1}{x})$, $q \in \mathbb{N}$, and since $\varphi(\frac{1}{x}) \not\in E_s$, $s = 1, 2, \ldots, q - 1$, $H-7$ does not hold.

**Example 3.** Let $X, Y$ be the same as in Example 1. Put $E_n = X^n \times Y^n$. Then $E = X^N \cap \bigcup \{E_n; n \in \mathbb{N}\}$ with the topology inherited from $X^N$. If $B$ is the closed unit ball in $X$, then $B^N \cap E$ is bounded in $E$ but not contained in any $E_n$. Hence $DS$ and $H-3 \& 7$ do not hold. Further $\overline{E_n^{E_{n+1}}} = E_{n+1}$ and $H-1 \& 4$ do not hold, either. On the other hand, $H-2 \& 8$ hold since the topology of $E_n$ is inherited from $E_{n+1}$.

**Example 4.** Put $W(x) = \sqrt{1 + x^2}$, $x \in (-\infty, \infty)$, and $E_n = \{f \in L^2(\mathbb{R}); \|f\|^2 = \int_{\mathbb{R}} |W^{-n}f|^2 dx < +\infty\}$. The norm $\| \cdot \|_n$ makes $E_n$ into a Hilbert space. Since the set $\mathcal{D}$ from Example 2 is dense in each $E_n$, we have $E_{n+p} = \mathcal{D} E_{n+p} \subset E_{n+p} E_{n+p} \subset \overline{E_n E}$ and $H-1, 2, 3, 4, 7$ do not hold. But, by Theorem 4 in [2], DST holds.

To show that $H-8$ does not hold, take $f_k = W^n x_{[-k,k]} \in E_n$ and put $B = \{f_k; k \in \mathbb{N}\}$. Then $\|f_k\|_n^2 = 2k$ and $B \subset E_n$. Further $\|f_k\|_{n+1}^2 \leq \pi$ and $B$ is bounded in $E_{n+1}$. If $H-8$ held $B$ would be bounded in $E_n$, by Lemma 2, which is not true.

**References**