DISCRETE GENERALIZED CESÀRO OPERATORS

H. C. RHALY, JR.

Abstract. For $|\lambda| \leq 1$, $A_\lambda^*$ is the operator defined formally on the Hardy space $H^2$ by

$$(A_\lambda^* f)(z) = - (\lambda - z)^{-1} \int_\lambda^z f(s) \, ds, \quad |z| < 1.$$ 

If $\lambda = 1$, then the usual identification of $H^2$ with $l^2$ takes $A_1$ onto the discrete Cesàro operator. Here we answer questions about boundedness, spectra, unitary equivalence, compactness, and subnormality for the operators $A_\lambda$.

The Cesàro operator $C_0$ acting on the Hilbert space $l^2$ of square-summable complex sequences $\{a_n\}_{n=0}^{\infty}$ is defined by $C_0\{a_n\} = \{b_n\}$ where $b_n = \sum_{j=0}^{n} a_j/(n+1)$, $n = 0, 1, 2, \ldots$. This operator was studied extensively in [1] where it was shown, among other things, that $C_0$ is bounded with $\|C_0\| = 2$ and spectrum $\{z: |1 - z| \leq 1\}$. In [4] it was proved that $C_0$ is a subnormal operator.

For $0 < |\lambda| \leq 1$ we define the operator $A_\lambda$ on $l^2$ by $A_\lambda\{a_n\} = \{c_n\}$ where $c_n = \sum_{j=0}^{n} \lambda^{-j} a_j/(n+1)$, $n = 0, 1, 2, \ldots$; also define $A_0$ by $A_0\{a_n\} = \{a_n/(n+1)\}$. Observe that $A_1 = C_0$. We identify $l^2$ isometrically with the Hardy space $H^2$ by sending $\{a_n\}_{n=0}^{\infty}$ onto $f(z) = \sum_{n=0}^{\infty} a_n z^n$. $A_\lambda$ then becomes an operator on $H^2$.

$A_\lambda^*$ can be expressed in closed form as follows. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $(A_\lambda^* f)(z) = \sum_{n=0}^{\infty} c_n z^n$ where $c_n = \sum_{k=0}^{n} \lambda^{-k} a_k/(n+1)$. Consider $\int_1^z f(s) \, ds$ where the path of integration is sufficiently nice. If $\lambda = 1$ and the path consists of two segments, one connecting 1 to 0 and the other connecting 0 to $z$, we have $\int_1^z f(s) \, ds = \int_1^0 f(s) \, ds - \int_0^1 f(s) \, ds$; the last integral exists by the Fejér-Riesz inequality [2, p. 46]. Integrating the Taylor series for $f$ term-by-term we have $\int_1^z f(s) \, ds = \sum_{n=0}^{\infty} a_n z^{n+1}/(n+1) - \sum_{n=0}^{\infty} a_n \lambda^{n+1}/(n+1)$. Comparing these Taylor coefficients with the Taylor coefficients of $(\lambda - z)(A_\lambda^* f)(z)$ we see that

$$(\lambda - z)(A_\lambda^* f)(z) = - \int_\lambda^z f(s) \, ds, \quad |z| < 1.$$ 

Hence

$$(A_\lambda^* f)(z) = - (\lambda - z)^{-1} \int_\lambda^z f(s) \, ds.$$ 

The author wishes to express his gratitude to T. L. Kriete III for suggesting the study of these integral operators.

Received by the editors August 15, 1981. 1980 Mathematics Subject Classification. Primary 47B99; Secondary 47A10, 47B05, 47B20. Key words and phrases. Cesàro operator, spectrum, compact operator, subnormal operator.

©1982 American Mathematical Society
0002-9939/82/0000-0199/$02.00

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
In matrix form, we have

\[
A^*_\lambda = \begin{bmatrix}
1 & \frac{\lambda}{2} & \frac{\lambda^2}{3} & \frac{\lambda^3}{4} & \cdots \\
0 & \frac{1}{2} & \frac{\lambda^2}{3} & \frac{\lambda^3}{4} & \cdots \\
0 & 0 & \frac{1}{3} & \frac{\lambda^2}{4} & \cdots \\
0 & 0 & 0 & \frac{1}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Since \( \|A_\lambda f\| \leq \|C_0(|f|)\| \), we see that \( A_\lambda \) is bounded with \( \|A_\lambda\| \leq 2 \). In order to get more information, we need some lemmas. The proof of Lemma 0 is found in [3].

**Lemma 0 (Schur Test).** If \( \alpha_{ij} \geq 0 \) (\( i, j = 0, 1, 2, \ldots \)), if \( p_i > 0 \) (\( i = 0, 1, 2, \ldots \)), and if \( \beta \) and \( \gamma \) are positive numbers such that

\[
\sum_{i} \alpha_{ij} p_i \leq \beta p_j \quad (j = 0, 1, 2, \ldots),
\]

\[
\sum_{j} \alpha_{ij} p_j \leq \gamma p_i \quad (i = 0, 1, 2, \ldots),
\]

then there exists an operator \( A \) on \( l^2 \) with \( \|A\|^2 \leq \beta \gamma \) and matrix \( \langle \alpha_{ij} \rangle \) (with respect to a suitable orthonormal basis).

**Lemma 1.** Assume \( 0 < \alpha < 1 \) and \( n \) is a positive integer. Define \( B_\alpha(n) = \langle \beta_{ij} \rangle_{i,j=1}^\infty \) by

\[
\beta_{ij} = \begin{cases}
\alpha^{i-j} / (i + 1) & \text{if } i \geq j + n, \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( B_\alpha(n) \) is bounded (on \( l^2 \)) and \( \|B_\alpha(n)\| \leq 2 \alpha^n \).

**Proof.** For \( \alpha > 0 \) we apply Lemma 0 with \( p_i = \alpha^i / \sqrt{i + 1} \). (The case \( \alpha = 0 \) is trivial.) For \( i = 0, 1, 2, \ldots, n - 1 \), we have

\[
\sum_j \beta_{ij} p_j = 0 = 0 p_i.
\]

For \( i \geq n \) we have

\[
\sum_j \beta_{ij} p_j = \sum_{j=0}^{i-n} \frac{\alpha^{i-j}}{i + 1} \cdot \frac{\alpha^j}{\sqrt{j + 1}} \leq \frac{\alpha^i}{i + 1} \int_0^{i+n+1} \frac{dx}{\sqrt{x}}
\]

\[
= \frac{\alpha^i}{i + 1} 2\sqrt{i + n + 1} \leq 2 \frac{\alpha^i}{\sqrt{i + 1}} = 2 p_i.
\]
For all $j$, we have
\[
\sum_i \beta_i p_i = \sum_{i=j+n}^{\infty} \frac{a_{i-j}}{i+1} \frac{a_i}{(i+1)^{3/2}} = \sum_{i=j+n}^{\infty} \frac{\alpha^{2i-j}}{(i+1)^{3/2}} \leq \alpha^{i+2n} \frac{1}{(j+n)^{3/2}} \frac{2\alpha^{j+2n}}{(j+n)^{3/2}} \leq 2\alpha^2 \sum_{i=j+n}^{\infty} \frac{\alpha^i}{(i+1)^{3/2}} = 2\alpha^2 p_j.
\]

It follows that $\|B_0(n)\| \leq 2(2\alpha^2 n) = (2\alpha^2)^2$.

**Theorem 1.** $A_\lambda$ is bounded for $|\lambda| \leq 1$ and $\|A_\lambda\| \leq 1 + 2 |\lambda|$.

**Proof.** We observe that $D \equiv A_{|\lambda|} - B_{|\lambda|}(1)$ is the diagonal operator with diagonal \{\frac{1}{n}\}_{n=1}^\infty. Since $\|A_\lambda\| \leq \|A_{|\lambda|}\|$, it follows from Lemma 1 that

\[
\|A_\lambda\| \leq \|B_{|\lambda|}(1) + D\| \leq 2|\lambda| + 1.
\]

**Proposition.** $A_\lambda$ is unitarily equivalent to $A_{|\lambda|}$, $0 < |\lambda| < 1$.

**Proof.** Consider the diagonal operator $D_\lambda$ with diagonal \{\frac{1}{n}\}_{n=1}^\infty. It is routine to check that $D_\lambda$ is unitary and $A_{|\lambda|} D_\lambda = D_\lambda A_\lambda$.

This result reduces our study to $A_\alpha$, $0 < \alpha < 1$. The diagonal operator $A_0$ is compact and Hermitian and has spectrum \{\{\frac{1}{n}\}_{n=1}^\infty \cup \{0\}. The Cesàro operator $A_1$ was studied in [1] and [4]. We shall now restrict ourselves to $0 < \alpha < 1$.

**Theorem 2.** The point spectrum of $A_\alpha$, $0 < \alpha < 1$, is the set \{\{\frac{1}{n}\}_{n=1}^\infty \cup \{0\}. The eigenvector corresponding to the simple eigenvalue \frac{1}{n} has closed form $f(z) = z^{-1}(1 - \alpha z)^{-n}$. The eigenvectors for $A_\alpha$ span $H^2$.

**Proof.** If $A_\alpha f = g$, then $f(0) = g(0)$, and if $n \geq 1$, then $f(n) = (n + 1)g(n) - \alpha g(n-1)$. Consequently, if $A_\alpha f = \gamma f$, then $f(n) = \gamma((n + 1)g(n) - \alpha g(n-1))$, or $(\gamma(n + 1) - 1)f(n) = \alpha g(n-1)$ for $n \geq 1$. If $m$ is the smallest integer for which $f(m) \neq 0$, then $\gamma = 1/(m+1)$, so $0 < \gamma < 1$. Thus $f(n) = 0$ for $n < m$ and $f(n) = \alpha g(n-1)/(n-m)$ for $n > m$. This implies that

\[
f(m + n) = \alpha^n \prod_{j=1}^n \left( \frac{m}{j} + 1 \right) f(m), \quad n \geq 1.
\]

We conclude from this that all the eigenvalues are simple. Since

\[
\frac{|f(m + n + 1)|^2}{|f(m + n)|^2} = \alpha^2 \left( \frac{m}{n+1} + 1 \right)^2 \to \alpha^2 \quad \text{as } n \to \infty,
\]

the ratio test implies that $f \in l^2$. Therefore $1/(m + 1)$ is a simple eigenvalue for $A$; from (1) we find that the corresponding eigenvector has the form

\[
f(z) = z^m + \sum_{n=1}^\infty \alpha^n \frac{(m + 1)(m + 2) \cdots (m + n)}{n!} z^{m+n}
\]

\[
= z^m (1 - \alpha z)^{-(m+1)}.
\]
Finally, assume \( g \in H^2 \) and \( g \) is orthogonal to all the eigenvectors of \( A_\alpha \). Then
\[
\frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) (e^{i\theta})^m (1 - \alpha e^{i\theta})^{-m-1} d\theta = 0 \quad \text{for all } m,
\]
so
\[
\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} g(e^{i\theta}) \psi_\alpha (e^{i\theta})^{m+1} d\theta = 0
\]
for all \( m \), where \( \psi_\alpha (z) = z(1 - \alpha z)^{-1} \). Since \( \psi_\alpha \) is analytic in \( D = \{ z : |z| < 1 \} \) and is continuous and univalent in \( \overline{D} \), it follows that for any nonnegative integer \( k \) there exists a sequence of polynomials \( \{ p_j \} \) such that \( p_j \circ \psi_\alpha \to z^k \) uniformly on \( \overline{D} \); as a consequence, \( \{ \psi_\alpha^n \}_{n=0}^\infty \) spans \( H^2 \) [5, p. 8]. Therefore \( zg(z) \) is constant in \( H^2 \). Hence \( g = 0 \). This completes the proof.

Before proceeding, we remark that in case \( \alpha < 1 \), \( A_\alpha \) can be shown to have the following closed form:
\[
(A_\alpha f)(z) = z^{-1} \int_0^z (1 - as)^{-1} f(s) ds, \quad |z| < 1.
\]

**Theorem 3.** The point spectrum of \( A_*^\alpha \), \( 0 < \alpha < 1 \), is the set \( \{ \frac{1}{\alpha} \}_{n=1}^\infty \). The eigenvector corresponding to the simple eigenvalue \( \frac{1}{\alpha} \) has closed form \( f(z) = (\alpha - z)^{n-1} \). The eigenvectors for \( A_*^\alpha \) span \( H^2 \).

**Proof.** Observe first that \( (A_*^\alpha f)(n) = \sum_{k=n}^{\infty} \alpha^{k-n} f(k)/(k + 1) \). If \( A_*^\alpha f = g \), then \( f(n) = (n+1)(g(n) - \alpha g(n + 1)) \) for \( n = 0, 1, 2, \ldots \). Consequently, if \( A_*^\alpha f = \gamma f \), then \( f(n) = \gamma(n+1)(f(n) - \alpha f(n + 1)) \). It follows that 0 is not an eigenvalue of \( A_*^\alpha \) (if \( \gamma = 0 \), then \( f(n) = 0 \) for all \( n \)). Therefore \( f(n + 1) = \alpha^{-1} [1 - 1/\gamma(n + 1)] f(n) \). This implies that if \( n \geq 1 \), then
\[
f(n)^2 = \alpha^{-2n} \prod_{j=1}^n \left[ 1 - \frac{1}{j\gamma} \right] f(0).
\]
From this we conclude that all the eigenvalues are simple. Assume \( \gamma \notin \{ \frac{1}{\alpha} \}_{n=1}^\infty \). Then \( |f(n+1)|^2/|f(n)|^2 \to \alpha^{-2} > 1 \) as \( n \to \infty \), and it follows from the ratio test that \( f \notin l^2 \). Now assume \( \gamma = \frac{1}{m} \), \( m \) a positive integer. Take \( f(0) = \alpha^{-m-1} \) in (2); then
\[
f(z) = \sum_{k=0}^{m-1} \binom{m-1}{k} \alpha^{m-1-k} (-z)^k = (\alpha - z)^{m-1},
\]
so \( f \in H^2 \). Hence \( \frac{1}{m} \) is an eigenvalue for \( A_*^\alpha \). The eigenvector corresponding to eigenvalue \( \frac{1}{m} \) is a polynomial of degree \( m-1 \); this makes it clear that the eigenvectors for \( A_*^\alpha \) span \( H^2 \).

**Theorem 4.** For \( 0 < \alpha < 1 \), \( A_\alpha \) is compact and \( \sigma(A_\alpha) \) (the spectrum of \( A_\alpha \)) is the set \( \{ \frac{1}{\alpha} \}_{n=1}^\infty \cup \{ 0 \} \). \( A_\alpha \) is not hyponormal (and hence not subnormal) if \( 0 < \alpha < 1 \).

**Proof.** Observe that
\[
\|A_\alpha - (A_\alpha - B_\alpha(n))\| = \|B_\alpha(n)\| \leq 2\alpha^n
\]
for \( n \) a positive integer by Lemma 1. Letting \( n \to \infty \) we see that \( A_\alpha \) is the norm limit of the sequence of compact operators \( A_\alpha - B_\alpha(n) \) and is therefore compact. Since
\( \sigma(A_{\alpha}) \) is closed and \( \{ \frac{1}{n} \}_{n=1}^{\infty} \subseteq \sigma(A_{\alpha}) \) (by Theorem 2), we must have \( 0 \in \sigma(A_{\alpha}) \). Since \( A_{\alpha} \) is compact we know that if \( \gamma \neq 0 \) and \( \gamma = \sigma(A_{\alpha}) \) then \( \gamma \in \pi_0(A_{\alpha}) \) (the point spectrum of \( A_{\alpha} \)) and \( \gamma \in \pi_0(A_{\alpha}^*) \) [6, p. 103]. It follows that \( \sigma(A_{\alpha}) = \{ \frac{1}{n} \}_{n=1}^{\infty} \cup \{ 0 \} \). If \( A_{\alpha} \) were hyponormal, it would be true that \( \| A_{\alpha} \| = r(A_{\alpha}) \) (the spectral radius of \( A_{\alpha} \)) [3, Problem 162]. We just determined that \( r(A_{\alpha}) = 1 \) if \( \alpha < 1 \). Since \( \| A_{\alpha} \| \geq \| A_{\alpha} 1 \| > 1 \), it is clear that \( A_{\alpha} \) cannot be hyponormal if \( 0 < \alpha < 1 \).

References


Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903

Current address: Department of Mathematics, University of Mississippi, University, Mississippi 38677