THE TRANSLATION INVARIANT UNIFORM APPROXIMATION PROPERTY FOR COMPACT GROUPS

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Abstract. We show that, unlike the abelian case, the translation invariant uniform approximation property fails in a strong way for $L^1$ of a compact connected semisimple Lie group.

1. We recall (see [5, 4 and 1]) that a Banach space $X$ is said to have the uniform approximation property if there are $k \geq 1$ and a sequence $q(n)$ of positive numbers such that for any $m$-dimensional subspace $E \subset X$ there exists an operator $T: X \to X$ for which $Tx = x$ for $x \in E$, $\|T\| \leq k$, $\dim TX \leq q(m)$.

As an important example we recall (see [4]) that the reflexive Orlicz spaces have the uniform approximation property. The purpose of this paper is to show that the situation is strongly different when we consider a Banach function space on a compact group and the translation invariant analogue of the uniform approximation property, i.e. if we modify the above definition assuming that $X$, $E$ and $T$ are translation invariant. This notion was introduced by Bożeiko and Pełczyński in [1], where the case of a compact abelian group was considered; they proved the theorem below and they obtained, as a consequence, a general result on the translation invariant analogue of the uniform approximation property for translation invariant function Banach spaces on $G$.

Theorem 1 (Bożeiko and Pełczyński). Let $G$ be a compact abelian group with dual group $\hat{\Gamma}$. For every $k > 1$ there exists a positive sequence $q_k(n)$ such that for every finite set $M \subset \Gamma$ there exists a trigonometric polynomial $g$ such that $g|_M = 1$, $\|g\|_1 \leq k$, $\mathrm{card}(\mathrm{supp}(g)) \leq q_k(\mathrm{card} M)$.

The above theorem is no longer true for arbitrary compact groups. We shall prove the following

Theorem 2. Let $G$ be a compact connected semisimple Lie group with dual object $\hat{G}$ (a maximal set of pairwise inequivalent unitary irreducible representations of $G$). Let $m \geq 1$, let $\{M_h\}_{h=1}^\infty$ be a sequence of pairwise disjoint subsets of $\hat{G}$ of cardinality at most $m$ and let $\{P_h\}_{h=1}^\infty$ be a sequence of trigonometric polynomials such that $\hat{P}_h(\sigma) = I_\sigma$ for all $\sigma \in M_h$ and $\mathrm{card}(\mathrm{supp}(\hat{P}_h)) \leq \text{const}_m$, then $\|P_h\|_1 \to \infty$ (as $h \to \infty$).

The author wishes to thank P. M. Soardi for raising the question considered here. In the next section we shall fix the notation.
2. Let $G$ denote a compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and $T$ a maximal torus with Lie algebra $\mathfrak{t}$. The complexification $\mathfrak{g}_C$ is a Cartan subalgebra of $\mathfrak{g}_C$ and we denote by $\Delta$ the set of roots of $(\mathfrak{g}_C, \mathfrak{t}_C)$. We choose in $\Delta$ a system $P$ of positive roots and the associated system of simple roots. We set $\beta = \frac{1}{2} \sum_{\alpha \in \rho} \alpha$. The weights of $G$ are ordered by letting $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1$ is a sum (possibly empty) of simple roots [6, p. 314]. We write $\lambda_1 < \lambda_2$ if $\lambda_1 \leq \lambda_2$ and $\lambda_1 \neq \lambda_2$. If $E$ is a set of weights of $G$, we say that $\lambda$ is a minimal (maximal) weight in $E$ if no weight $\gamma$ in $E$ satisfies $\gamma < \lambda$ ($\gamma > \lambda$). For every weight $\lambda$ the symmetric sum $S(\lambda)$ is defined on $T$ by $S(\lambda)(t) = \sum \exp \mu(u)$, where $t = \exp u$ ($u \in t$) and the summation runs over all $\mu$ in the orbit of $\lambda$ under the action of the Weyl group $W$; we denote by $\tilde{S}(\lambda)$ the unique central continuous extension of $S(\lambda)$ to the whole of $G$. We recall that the dual object $\hat{G}$ may be identified with the semilattice $\Sigma$ of the dominant weights of $G$; throughout this paper we shall use the same symbol to denote a representation in $\hat{G}$ and his highest weight in $\Sigma$. Finally, for every $\lambda$ in $\hat{G}$, $\chi_\lambda$ and $d_\lambda$ denote the character and the dimension of $\lambda$ respectively, while $I_\lambda$ is the identity operator on the Hilbert space $H_{d_\lambda}$.

3. We need a lemma, whose proof is contained in [2].

**Lemma.** For any $\lambda$ in $\Sigma$ the function $\tilde{S}(\lambda + 2\beta)$ has the following Fourier expansion (on $G$): $\tilde{S}(\lambda + 2\beta) = \chi_{\lambda+2\beta} + \sum m_\gamma x_\gamma + \tau \chi_\lambda$, where $\tau = \pm 1$, the $m_\gamma$ are relative integers and $\lambda < \gamma < \lambda + 2\beta$ for any $\gamma$ in the above summation.

**Proof of Theorem 2.** We choose a maximal weight $\lambda_h$ in $M_h$ and we write $F_h = \text{supp} (\tilde{S}(\lambda_h + 2\beta)) \cap \text{supp}(\tilde{P}_h)$; we have two cases:

- (a') $F_h = \{\lambda_h\}$,
- (b') $F_h = \{\lambda_h, \gamma_1, \ldots, \gamma_r\}$ ($r \geq 1$).

In the first case we obtain

$$\|P_h\|_1 \geq \frac{1}{\|\tilde{S}(\lambda_h + 2\beta)\|_\infty} \cdot \|P_h * \tilde{S}(\lambda_h + 2\beta)\|_\infty \geq \frac{1}{\text{card} W} \cdot \|\chi_{\lambda_h}\|_\infty \to \infty \quad (\text{as } h \to \infty).$$

In the case (b') we first observe that (by the lemma and the maximality of $\lambda_h$) we have $M_h \cap \{\gamma_1, \ldots, \gamma_r\} = \emptyset$. Then we choose a minimal weight $\tilde{\gamma}_1$ in $\{\gamma_1, \ldots, \gamma_r\}$ and an integer $n_1$ such that the polynomial $T_1 = \tau \tilde{S}(\lambda_h + 2\beta) + n_1 \tilde{S}(\tilde{\gamma}_1 + 2\beta)$ satisfies $T_1(\gamma'_1) = 0$; observe that $T_1(\lambda_h) = d^{-1}_{\lambda_h} I_{\lambda_h}$, otherwise, by the lemma, we should have $\lambda_h < \tilde{\gamma}_1$ and $\tilde{\gamma}_1 < \lambda_h$; in the same way one verifies that $T_1(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in $M_h$. Then we choose a minimal weight $\tilde{\gamma}_2$ in $\{\gamma_1, \ldots, \gamma_r\} \setminus \{\tilde{\gamma}_1\}$ and an integer $n_2$ such that $T_2 = T_1 + n_2 S(\tilde{\gamma}_2 + 2\beta)$ satisfies $T_2(\tilde{\gamma}_2) = 0$; observe that $T_2(\lambda_h) = d^{-1}_{\lambda_h} I_{\lambda_h}$, while $T_2(\tilde{\gamma}_2) = 0$ (we cannot have $\tilde{\gamma}_2 \leq \tilde{\gamma}_1$, hence, by the lemma, $\tilde{\gamma}_2 \notin \text{supp}(\tilde{S}(\tilde{\gamma}_2 + 2\beta))$);

moreover, as above, $T_2(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in $M_h$. Now we choose a minimal weight $\tilde{\gamma}_3$ in $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_2\} \setminus \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ and an integer $n_3$ such that $T_3 = T_2 + n_3 S(\tilde{\gamma}_3 + 2\beta)$ satisfies $T_3(\tilde{\gamma}_3) = 0$; arguing as above we observe that $T_3(\lambda_h) = d^{-1}_{\lambda_h} I_{\lambda_h}$, $T_3(\gamma_i) = 0$ ($i = 1, 2$), $T_3(\lambda) = 0$ for any $\lambda \neq \lambda_h$ in $M_h$. We go on until we
construct a trigonometric polynomial \( T^1 = T^1_i \) with the following properties: \( \hat{T}^1(\lambda_h) = d_{\lambda_h}^{-1}I_{\lambda_h} \); \( \hat{T}^1(j_i) = 0 \) for all \( j = 1, \ldots, r \); \( \hat{T}^1(\lambda) = 0 \) for any \( \lambda \neq \lambda_h \) in \( M_h \); \( \| T^1 \|_\infty \leq \text{const}_\mu \) (in particular \( \| T^1 \|_\infty \) does not depend upon \( h \)).

Now we write \( F_2 = \text{supp}(\hat{T}^1) \cap \text{supp}(\hat{P}_h) \); we have two cases:

(a\(^2\)) \( F_2 = \{ \lambda_h \} \),

(b\(^2\)) \( F_2 = \{ \lambda_h, \gamma_1^2, \ldots, \gamma_2^2 \} (s \geq 1) \)

(observe that, by construction, \( \{ \gamma_1^2, \ldots, \gamma_2^2 \} \cap \{ \gamma_1^2, \ldots, \gamma_2^2 \} = \emptyset \); observe also that for any \( \gamma_2^2 \) we have \( \lambda_h < \gamma_2^2 \), hence we cannot have \( \gamma_2^2 < \lambda \) for any \( \lambda \) in \( M_h \).) In the case (a\(^2\)) we obtain, as above,

\[
\| P_h \|_1 \geq \frac{1}{\| T^1 \|_\infty} \cdot \| P_h \ast T^1 \|_\infty \geq \text{const}_\mu \cdot d_{\lambda_h} \rightarrow \infty \quad (h \rightarrow \infty).
\]

In the case (b\(^2\)) we argue exactly as above (taking the set \( \{ \gamma_1^2, \ldots, \gamma_2^2 \} \) in place of \( \{ \gamma_1^2, \ldots, \gamma_2^2 \} \) until we construct a trigonometric polynomial \( T^2 \) which has the following properties: \( \hat{T}^2(\lambda_h) = d_{\lambda_h}^{-1}I_{\lambda_h} \); \( \hat{T}^2(j_i) = 0 \) for all \( j = 1, \ldots, r \); \( \hat{T}^2(\gamma_2^2) = 0 \) for all \( j = 1, \ldots, s \); \( \| T^2 \|_\infty \leq \text{const}_\mu \).

Then we write \( F_3 = \text{supp}(\hat{T}^2) \cap \text{supp}(\hat{P}_h) \); again we have two cases (a\(^3\)) and (b\(^3\)) and we go on. To complete the proof we recall that \( \text{card}(\text{supp}(\hat{P}_h)) \leq \text{const}_m \), hence we cannot be in the second case for more than \( \text{const}_m \) steps.

REMARK. In [3] it was shown that \( \| \chi_\sigma \|_3 \rightarrow \infty \) as \( \sigma \) runs in \( \Sigma \). Hence the technique of the above proof shows that not only \( \| P_h \|_1 \rightarrow \infty \), but also the \( (L^p, L^p) \) convolutor norm of \( P_h \) diverges for \( 1 \leq p < \frac{3}{2} \) or \( p \geq 3 \). We omit the details.

REFERENCES


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