TOTALLY REAL MINIMAL SUBMANIFOLDS
IN A COMPLEX PROJECTIVE SPACE

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Abstract. We give a pinching theorem with respect to the scalar curvatures of
4-dimensional conformally flat totally real minimal submanifolds in a 4-dimensional
complex projective space.

1. Introduction. Among all submanifolds of an almost Hermitian manifold, there
are two typical classes: one is the class of holomorphic submanifolds and the other is
the class of totally real submanifolds. There have been many results in the theory of
holomorphic submanifolds; on the other hand, there have been only a few results in
the theory of totally real submanifolds.

H. Naitoh [2], M. Takeuchi [3] classified submanifolds in a real and complex space
form with parallel second fundamental form.

Among such examples, there exist three n-dimensional conformally flat totally real
minimal submanifolds in a complex projective space $P_n$ of constant holomorphic
sectional curvature 4:

(i) a totally geodesic submanifold,
(ii) a flat torus,
(iii) a Riemannian product of $S^1(\sin a \cos a)$ and $S^{n-1}(\sin a)$, where $S^r(r)$ is an
$n$-dimensional sphere with radius $r$ and $\tan a = \sqrt{n}$.

The purpose of this paper is to give a characterization of (ii) and (iii) of 4
dimension.

Theorem. Let $M$ be a 4-dimensional compact orientable conformally flat totally real
minimal submanifold in $P_4$. If $M$ has nonnegative Euler number and the scalar
curvature $\rho$ of $M$ is between 0 and 15/2, then $\rho$ is 0 or 15/2 and $M$ is (ii) ($\rho = 0$),
(iii) ($\rho = 15/2$) or its covering spaces.

Remark. If $n = 4$, B. Y. Chen and K. Ogiue’s result [1] implies that every compact
totally real minimal submanifold in $P_4$ with $\rho \geq 64/7$ is totally geodesic ($\rho = 12$).
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2. Proof of Theorem. We use the same notations and terminologies as in [1]. It was
proved in [1] that the second fundamental form of the immersion satisfies

\[
\frac{1}{2} \Delta \| \sigma \|^2 = \| \nabla' \sigma \|^2 + \sum_{i,j} \text{tr}(A_i \ast A_j - A_j \ast A_i)^2 - \sum_{i,j} (\text{tr} A_i \ast A_j)^2 + 5 \| \sigma \|^2.
\]

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Since
\[ \sum_{i,j} \text{tr} (A_i^* A_j - A_j^* A_i)^2 = - \sum_{i,j,k,l} \left( \sum_m (h^*_{ikm} h_{ljm} - h^*_{ikm} h_{ljm}) \right)^2, \]
this, together with the equation of Gauss, implies
\[ \sum_{i,j} \text{tr} (A_i^* A_j - A_j^* A_i)^2 = \| R \|^2 + 4\rho - 24. \]
By the same argument as above, we have
\[ \sum_{i,j} (\text{tr} A_i^* A_j)^2 = \| S \|^2 - 6\rho + 36. \]
Combining (1) with (2) and (3), we obtain
\[ \| A \|^2 = \| V' \|^2 - \| R \|^2 - \| S \|^2 + 5\rho. \]
From the assumption that \( M \) is conformally flat, we obtain
\[ \| R \|^2 - 2\| S \|^2 + \frac{1}{2} \rho^2 = 0, \]
which, together with (4), asserts
\[ \frac{1}{2} \Delta \| \sigma \|^2 = \| \nabla' \sigma \|^2 - \| R \|^2 - \| S \|^2 + \frac{1}{2} \rho^2 + 5\rho. \]
Taking the integrals of both sides of it and using Green's theorem, we have
\[ \int_M \| \nabla' \sigma \|^2 * l_M = \int_M \left\{ \left( \frac{3}{2} \| S \|^2 - \frac{1}{2} \rho^2 - 5\rho \right) * l_M \right\}. \]
On the other hand, by the Gauss-Bonnet theorem, the Euler number \( \chi(M) \) of \( M \) is given by
\[ \chi(M) = \frac{1}{32\pi^2} \int_M \left\{ \| R \|^2 - 4\| S \|^2 + \rho^2 \right\} * l_M. \]
It follows from (5) that
\[ \chi(M) = \frac{1}{32\pi^2} \int_M \left\{ \frac{2}{3} \rho^2 - 2\| S \|^2 \right\} * l_M. \]
Combining (6) with (7), we get an integral formula:
\[ 48\pi^2 \chi(M) + \int_M \| \nabla' \sigma \|^2 * l_M = \int_M \frac{2}{3} \rho \left\{ \rho - \frac{15}{2} \right\} * l_M. \]
Theorem follows from the integral formula and results in [2, 3].

REFERENCES

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