LOWER BOUNDS FOR THE UNKNOTTING NUMBERS OF CERTAIN TORUS KNOTS

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ABSTRACT. In this paper we shall show that the unknotting numbers of the \((l, 2kl \pm 1)\)-torus knots are at least \((k(l^2 - 1) - 2)/2\) for \(l\) odd and \((kl^2 - 2)/2\) for \(l\) even, where \(l\) is an integer greater than one and \(k\) is a positive integer.

1. Introduction. Unless otherwise stated all manifolds and maps are smooth and all knots and links are in \(S^3\).

The unknotting number of a knot is the minimum number of crossings which must be changed to make the knot trivial. Let \(l\) and \(m\) be integers. The \((l, m)\)-torus link is the link which lies on an unknotted torus and sweeps around it \(l\) times in the longitude and \(m\) times in the meridian. When \(l\) and \(m\) are relatively prime, it is a knot and called the \((l, m)\)-torus knot. Milnor [3] conjectured that the unknotting number of a torus knot is equal to the genus of it. It is well known that the genus of the \((l, m)\)-torus knot is equal to \((l - 1)(m - 1)/2\). It is not hard to see that the unknotting number of a torus knot is at most the genus of it. In this paper we shall show the following.

THEOREM A. Let \(l\) be an integer greater than one and \(k\) a positive integer. Then the unknotting numbers of the \((l, 2kl \pm 1)\)-torus knots are at least \((k(l^2 - 1) - 2)/2\) if \(l\) is odd, and \((kl^2 - 2)/2\) if \(l\) is even.

Murasugi [4] showed that the unknotting number of the \((2, m)\)-torus knot is equal to the genus of it. Weintraub [8] showed that the unknotting number of the \((m - 1, m)\)-torus knot is at least \((m^2 - 5)/4\) if \(m\) is odd, and \((m^2 - 4)/4\) if \(m\) is even.

The author wishes to thank Professors Mitsuyoshi Kato and Hiroshi Noguchi for helpful suggestions.

2. Preliminaries. The following is a theorem of Rohlin [5], Hsiang-Szczarba [9], Thomas-Wood [6] and Weintraub [7].

THEOREM 1. Let \(N\) be an oriented, connected, simply connected, closed 4-manifold. Let \(M\) be an oriented, connected, closed surface embedded in \(N\). Suppose that \(M\) represents a 2-homology class \([M]\) of \(H_2(N; Z)\) and that \([M]\) is divisible by a positive integer \(d\) in the free abelian group \(H_2(N; Z)\). Let \(g_M\) be the genus of \(M\). Then

\[
2g_M \geq \frac{[M]^2}{d^2} \frac{d^2 - 1}{2} - \text{rank} H_2(N; Z) - \text{signature} N
\]

for \(d\) odd, and similarly for \(d\) even with \(d^2/2\) instead of \((d^2 - 1)/2\), where \([M]^2\) is the self-intersection number of \(M\).

Received by the editors August 26, 1981.

1980 Mathematics Subject Classification. Primary 57M25.

Key words and phrases. Torus knot, torus link, unknotting number of a knot, genus of a knot.

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The following theorem originally due to Boardman [1] and Weintraub [8].

**THEOREM 2.** Let $N$ be an oriented 4-manifold, and let $N_0 = N - \text{Int } B^4$, where $B^4$ is an embedded 4-ball. Suppose $\alpha \in H_2(N_0, \partial N_0; \mathbb{Z})$ is represented by an embedding $\Phi : (B^2, S^1) \rightarrow (N_0, \partial N_0)$ and let $K$ denote the knot given by $\Phi|S^1 : S^1 \rightarrow \partial N_0 = S^3$. If $u$ is the unknotting number of $K$, then $\alpha$ is represented by an embedded 2-sphere in $N \# (uW)$, where $W = CP^2 \# (-CP^2)$.

The proof is given by using $W$ instead of $CP^2$ and $-CP^2$ of Theorem 7 of [8].
3. Proof of Theorem A. Let $\alpha$ and $\beta$ be the generators of

$$H_2(S^2 \times S^2 - \text{Int} B^4, \partial(S^2 \times S^2 - \text{Int} B^4); \mathbb{Z}),$$

and let $l$ and $m$ be positive integers. Then $l\alpha + m\beta$ can be represented by an embedded disk, say $D_{l,m}$. Let $K_{l,m}$ be the knot which is the boundary of $D_{l,m}$. 
Lemma. Let $u$ be the unknotting number of $K_{i,m}$. Suppose that $l$ and $m$ are divisible by a positive integer $d$. Then

$$u \geq \begin{cases} \frac{lm(d^2 - 1)}{2d^2} - 1 & \text{if } d \text{ is odd,} \\ \frac{lm}{2} - 1 & \text{if } d \text{ is even.} \end{cases}$$

Proof. By Theorem 2, $la + m\beta$ is represented by an embedded 2-sphere in $(S^2 \times S^2) \# (uW)$. Then, by Theorem 1,

$$0 \geq \frac{2lm}{d^2} \left( \frac{d^2 - 1}{2} - 2(1 + u) \right)$$

and

$$0 \geq \frac{2lm}{d^2} \left( \frac{d^2 - 2(1 + u)}{2} \right)$$

and elementary algebra yields the lemma.

Let $l$ be an integer greater than one, and let $k$ be a positive integer. We show that the class

$$la + k\beta \in H_2(S^2 \times S^2 - \text{Int } B^4, \partial(S^2 \times S^2 - \text{Int } B^4); Z)$$

can be represented by an embedded disk with boundary the $(l, 2kl \pm 1)$-torus knot. The class $la + k\beta$ can be represented by $l + kl$ embedded disks with boundary the link $L$ of $l + kl$ components illustrated in Figure 1. Let $K_1, \ldots, K_l, K'_1, \ldots, K'_{kl}$
The (4,8)-torus link

**Figure 8**

be as in Figure 1. Then $l\alpha + kl\beta$ can be represented by the disk obtained by connecting $K_1, \ldots, K_l, K'_1, \ldots, K'_k$ by $l + kl - 1$ strips in $\partial(S^2 \times S^2 - \text{Int} B^4) = S^3$ (see Kervaire–Milnor [2]). We connect $K_i$ and $K'_{(i-1)l+i}$ by the strip $E_{i,j}$ as in Figure 2 for $i = 1, \ldots, l$ and $j = 1, \ldots, k$, thereby obtaining a collection of $l$ disks representing the class $l\alpha + kl\beta$, whose boundaries form a link of $l$ components.
Let $L'$ be the link which consists of the boundaries of the disks obtained by the above construction. We show that the link $L'$ is the $(l, 2kl)$-torus link as follows; it is sufficient to show that there is an isotopy of $S^3$ which deforms the part of $L'$ contained in the 3-ball $B$ in Figure 2 into the part of the $(l, 2l)$-torus link contained in the 3-ball $B'$ in Figure 3 and which is relative to the complement of $B$. In Figures 4, 5, 6 and 7 we illustrate only the parts contained in the 3-balls, and all isotopies are relative to the complements of the 3-balls. We may consider that the components $K_2, \ldots, K_l, (K'_2, \ldots, K'_l)$ of $L'$ are contained in a narrow tube $T$ (resp. $T'$) as in Figure 4. We connect $K_1$ and $K'_1$ by the strip $E_{1,1}$. Then we have the link which is isotopic to the link illustrated in Figure 5. We take $K_2 (K'_2)$ out of $T$ (resp. $T'$) and connect $K_2$ and $K'_2$ by the strip $E_{2,1}$. Then we have the link which is isotopic to the link as in Figure 6, where the components $K_1 \# K'_1$ and $K_2 \# K'_2$ form the $(2, 4)$-torus link. We suppose that we have the link as in Figure 7 after connecting $K_i$ and $K'_i$ by the strip $E_{i,1}$ for $i = 1, \ldots, r - 1$ $(2 \leq r \leq l - 1)$ as above and that the components $K_1 \# K'_1, K_2 \# K'_2, \ldots, K_{r-1} \# K'_{r-1}$ form the $(r - 1, 2(r - 1))$-torus link. We connect $K_r$ and $K'_r$ by the strip $E_{r,1}$ as in Figure 7. Then we have the link which is isotopic to the similar link as in Figure 7 with $K_{r-1} \# K'_{r-1}, K_r \# K'_r, K_{r+1}$ and $K'_{r+1}$ instead of $K_{r-2} \# K'_{r-2}, K_{r-1} \# K'_{r-1}, K_r$ and $K'_r$. Then we have inductively the link $(K_1 \# K'_1) \cup \cdots \cup (K_l \# K'_l)$ which is isotopic to the $(l, 2l)$-torus link by the composite of the above isotopies as required. Therefore we have shown that the link $L'$ is the $(l, 2kl)$-torus link.

The $(l, 2kl)$-torus link is represented as in Figure 8. Let $K''_i$ be the component $K_i \# K'_i \# K''_{i+1} \# \cdots \# K''_{(k-1)+1}$ of $L'$. We connect $K''_i$ and $K''_{i+1} (i = 1, \ldots, l - 1)$ by the strip $E''_{i,1}$ as in Figure 9. Then we have the $(l, 2kl - 1)$-torus knot. When we connect $K''_i$ and $K''_{i+1} (i = 1, \ldots, l - 1)$ by the strip $E''_{i,1}$ as in Figure 10, we have the $(l, 2kl + 1)$-torus knot. By Lemma, we obtain Theorem A.

**REFERENCES**


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