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HEIGHT ONE DIFFERENTIAL IDEALS IN POLYNOMIAL RINGS

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ABSTRACT. Let D be a derivation of a polynomial ring $k[X_1, \ldots, X_n]$ with k a field of characteristic 0 and $Dk = \{0\}$. If infinitely many principal prime ideals (f) satisfy $Df \in (f)$, then every maximal ideal contains such an (f).

1. Introduction. Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure, $R = k[X_1, \ldots, X_n]$ a polynomial ring in n variables ($X_1, \ldots, X_n$ will often be abbreviated as X), and $D = q_1\partial/\partial X_1 + \cdots + q_n\partial/\partial X_n$ ($q_i \in R$) a derivation of R/k. Call an ideal $I \subset R$ *differential* if $DI \subset I$. Intuitively, we think of D as a vector field on k^n, and an ideal I is differential if D induces a derivation on R/I, i.e. the vector field restricts to a (tangent) vector field on the zero locus $V(I)$. In [S], some basic facts concerning differential ideals are proved. In particular, by [S, p. 24, Theorem 1], if I is differential then so are the associated primes of I, so we may as well consider the irreducible components of $V(I)$.

We wish to study the existence of differential polynomials f such that $Df = hf$ ($h, f \in R$). If D induces tangent vector fields on infinitely many irreducible hypersurfaces, it is reasonable to expect that such hypersurfaces "fill" k^n, i.e. every point is contained in a differential hypersurface. The proof that this is so is the main result of this work. We then consider several derivations simultaneously. Finally, some examples and connections with other work are presented.

2. Main result. We state some theorems with the objective of proving Theorem 6. First, it is easy to see directly that if $f \in R$ is differential then so is each factor of f. Now,

THEOREM 1. *Suppose F, G are nonzero polynomials such that $DF = hF$, $DG = hG$, and $(F) \neq (G)$. Then every maximal ideal of R contains a height 1 differential prime ideal.*

PROOF. Any maximal ideal $m \subset R$ is the contraction of a maximal ideal $M = (X_1 - a_1, \ldots, X_n - a_n) \subset \bar{k}[X]$, since $\bar{k}[X]$ is integral over R. If $F(a) = 0$ (here $a = (a_1, \ldots, a_n)$) then $F \in m = M \cap R$. Otherwise, there exists $c_1 \in \bar{k}$ such that $(G - c_1F)(a) = 0$. Let $\{c_1, \ldots, c_i\}$ be the conjugates of c_1 over k, and set $H = (G - c_1F)(G - c_2F) \cdots (G - c_iF)$. $H(a) = 0$ and $H \in R$, so $H \in m$, and $DH = i \cdot hH$. In either case m contains a nonzero differential polynomial and at least one irreducible factor of the polynomial must be contained in m. $\square$

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COROLLARY. *If F, G as above exist and $[\bar{k}: k] \leqslant i$ then every maximal ideal of R contains a differential polynomial of degree $\leqslant i \cdot \max(\deg F, \deg G)$.* $\square$

The next theorem gives a condition for F, G to exist.

THEOREM 2. *Let f_i be nonzero irreducible polynomials with $Df_i = h_i f_i$ and $(f_i) \neq (f_j)$ for $i \neq j$. Suppose the h_i are linearly dependent over the rational numbers $\mathbf{Q}$, say*

$$m_1 h_1 + \cdots + m_r h_r - n_1 h_{r+1} - \cdots - n_s h_{r+s} = 0$$

where the m_i, n_i are nonnegative integers. Define $F = f_1^{m_1} \cdots f_r^{m_r}$, $G = f_{r+1}^{n_1} \cdots f_{r+s}^{n_s}$ ($G = 1$ if $s = 0$), $h = m_1 h_1 + \cdots + m_r h_r$. Then F, G are nonzero polynomials with no common factors and $DF = hF$, $DG = hG$.

PROOF. Compute DF, DG. $\square$

We will want to simplify to the case where k is a finitely generated extension of $\mathbf{Q}$. So, let K be a pure transcendental extension of k; $K = k(T_\alpha)$, where $\{T_\alpha\}$ is algebraically independent over k. Extend D to a derivation of $K[X]$ ($= K[X_1, \ldots, X_n]$) by setting $DK = \{0\}$.

THEOREM 3. *If a nonzero ideal $(f) \subset K[X]$ satisfying $(f) \cap R = (0)$ is differential then R contains infinitely many distinct irreducible differential polynomials.*

PROOF. Write $f = \Sigma(g_j/h_j)N_j$, where g_j, $h_j \in K_0 = k[T_\alpha]$, and the N_j are distinct monomials in the X_i. After dividing f by an element of K, we may assume that a monomial N of highest total degree appears with coefficient $= 1$ in f. k is infinite, so there are points (t_α) with coordinates in k such that $h_j(t_\alpha) \neq 0$ for all j. Let $K_{t_\alpha} = K_{0_{(T_\alpha - t_\alpha)}}$ be the localization of K_0 at the maximal ideal $(T_\alpha - t_\alpha)$, and let $R_{t_\alpha} = K_{t_\alpha}[X]$. By construction $f \in R_{t_\alpha}$, so $Df \in R_{t_\alpha}$. Since the coefficient of N is 1, f is not divisible (in R_{t_α}) by any nonunit of K_{t_α}.

By hypothesis, we have $Df = hf$ for some $h \in K[X]$. Write $h = h_1/h_2$, where $h_1 \in R_{t_\alpha}$ and $h_2 \in K_{t_\alpha}$. We see from $h_2 Df = h_1 f$ and unique factorization in R_{t_α} that h_2 divides h_1, hence $h \in R_{t_\alpha}$.

Define a k-homomorphism H_{t_α}: $R_{t_\alpha} \to R$ by $T_\alpha \mapsto t_\alpha$. Let $\bar{f} = H_{t_\alpha}(f)$ denote the image of f; then $\bar{f}$ is differential, because $D\bar{f} = \overline{(Df)} = \bar{h}\bar{f}$, and $\bar{f} \neq 0$ since $N \in R$. Each prime factor of $\bar{f}$ in R is differential. Let (t'_α) be another point with all $h_j(t'_\alpha) \neq 0$. If, for some j, $(g_j/h_j)(t_\alpha) \neq (g_j/h_j)(t'_\alpha)$, then $H_{t_\alpha}(f)$ and $H_{t'_\alpha}(f)$ are not associates, since $H_{t_\alpha}(N) = H_{t'_\alpha}(N) = N$. By hypothesis, some $g_j/h_j \notin k$ and k is infinite, so there exist infinitely many $H_{t_\alpha}(f) \in R$ which are not associates of each other. Since $\deg H_{t_\alpha}(f) = \deg f$ is constant, there must be infinitely many distinct, nonassociate, irreducible polynomials which are factors of $H_{t_\alpha}(f)$ for various (t_α); these irreducible factors are differential. $\square$

The following theorem was shown to us by Seidenberg.

THEOREM 4 (SEIDENBERG). *Suppose A is an integrally closed Noetherian domain with quotient field k, and the q_i in $D = q_1\partial/\partial X_1 + \cdots + q_n\partial/\partial X_n$ are elements of $A[X]$. If $h \in R$ satisfies $Df = hf$ for some nonzero $f \in R$, then $h \in A[X]$.*

PROOF. h does not change when f is multiplied by a nonzero element of A, so we may assume that $f \in A[X]$. Write $h = h_1/h_2$ where $h_1 \in A[X]$, $h_2 \in A$. A is an intersection of discrete valuation rings ($A = \bigcap A_p$, p the height 1 primes of A) and h_2 is contained in only finitely many p's, say $p_1, \ldots, p_l$. Let v_i be the associated valuations. Then there are elements $t_i \in k$ such that $v_i(t_j) = \delta_{ij}$ [**ZS**, p. 45, Theorem 18]. Extend v_i to a valuation of R by defining $v_i(g) = \min_j v_i(g_j)$, g_j the coefficients of $g \in R$. Set $m_i = v_i(f)$ (so $m_i \geqslant 0$) and $t = t_1^{m_1} \cdots t_l^{m_l}$. Then $v_i(f/t) = 0$. There are only a finite number of discrete valuations w_j of k whose rings contain A which, when extended to R satisfy $w_j(f/t) < 0$. Pick $a \in A$ with $w_j(a) > 0$ for all j and $v_i(a) = 0$ for all i. Set $f_1 = a^r f/t$, $r \gg 0$. Then for all discrete valuations w_p (corresponding to A_p) we have $w_p(f_1) \geqslant 0$. Hence $f_1 \in A[X]$ and furthermore $v_i(f_1) = 0$. Now for $w_p \notin \{v_i\}$ we have $w_p(h_1/h_2) \geqslant 0$. Also,

$$v_i(h_1/h_2) = v_i(Df_1) - v_i(f_1) = v_i(Df_1) \geqslant 0.$$

Hence $h_1/h_2 \in (\bigcap A_p[X]) = A[X]$. $\square$

The next theorem is the key observation leading to Theorem 6.

THEOREM 5. *If k is a finitely generated extension of a subfield k_0 then the h_i appearing in $Df_i = h_i f_i$ ($f_i \neq 0$) span a finite-dimensional k_0-vector space.*

PROOF. The proof is immediate if the h_i span a finite-dimensional vector space over a subfield of k which is an algebraic (and necessarily finitely generated) extension of k_0. Therefore we assume that k_0 is algebraically closed in k.

Let $(v) = (v_1, \ldots, v_r)$ be a transcendence basis of k/k_0. Take A to be the integral closure of $k_0[v]$ in k. Multiplying D by a nonzero element of A does not change the dimension of the k_0-vector space spanned by the h_i, so we may assume that the q_j (in $D = \sum q_j \partial/\partial X_j$) are in $A[X]$. By Theorem 1, the h_i are in $A[X]$.

Regard A as the coordinate ring of an affine piece of a normal projective variety (defined over k_0 and having function field k). The functions $v_1, \ldots, v_r$ have poles on a finite number of codimension 1 subvarieties W_i; let w_i be the valuations associated with the W_i. Since A is integral over $k_0[v]$, the W_i are the only codimension 1 subvarieties for which a function in A can have a pole.

Extend the w_i to $k[X]$ by defining $w_i(f) = \min_j w_i(f_j)$, f_j the coefficients of f. We have

$$w_i(h) + w_i(f) = w_i(Df) \geqslant \min_j \left(w_i\left(q_j f_{X_j}\right)\right) \geqslant N_i + w_i(f)$$

where $N_i = \min_j(w_i(q_j))$, so $w_i(h) \geqslant N_i$. Define a divisor $\Delta = \sum N_i W_i$. Then the coefficients of h are contained in the linear system $L(-\Delta) = \{g \in k\colon (g) \geqslant \Delta\}$, which is finite-dimensional over k_0 [**L**, p. 159, Theorem 2, p. 174, Theorem 5] (note that $\bar{k}_0$ is linearly disjoint from k over k_0). Taking into account the bounded degree (in X) of h, we conclude that $\{h_i\}$ is contained in a finite-dimensional k_0-vector space. $\square$

The main result is:

THEOREM 6. *Let k be a field of characteristic* 0, $R = k[X_1, \ldots, X_n]$ *a polynomial ring in n variables, and* $D = q_1\partial/\partial X_1 + \cdots + q_n\partial/\partial X_n$ $(q_i \in R)$ *a derivation of R/k. If infinitely many principal prime ideals (f) satisfy $Df \in (f)$, then every maximal ideal of R contains an irreducible nonzero element f with $Df \in (f)$.*

PROOF. Assume that infinitely many principal prime ideals (f) satisfy $Df \in (f)$. Let k_1 be the finitely generated extension of $\mathbf{Q}$ obtained by adjoining the coefficients of the q_i. Choose a transcendence basis (T_α) of k/k_1, and define $k_2 = k_1(T_\alpha)$. Then R is integral over $R_2 = k_2[X]$, so R_2 contains infinitely many prime ideals (f) with $Df \in (f)$. Let $R_1 = k_1[X]$. If $(f) \subset R_2$ is a height 1 prime, then $(f) \cap R_1$ has height 0 or 1 [**ZS**, p. 223, Theorem 35]. By the same theorem, if $(f_1) \cap R_1 = (f_2) \cap R_1$ are height 1, then $(f_1) = (f_2)$. So either R_1 contains infinitely many height 1 differential primes $(f) \cap R_1$, or some $(f) \cap R_1 = (0)$. In the latter case, Theorem 3 is applicable, and again R_1 contains infinitely many height 1 differential primes. Now apply Theorems 5, 2, and 1 (note that F, $G \in R_1$, $F/G \notin k_1$ implies that $F/G \notin k$). $\square$

3. Several derivations. Let $T = \{D_1, \ldots, D_l\}$ $(D_i = \Sigma\, q_{ij}\partial/\partial X_j)$ be a finite set of derivations of R/k. Call an ideal $I \subset R$ *T-differential* if $D_i I \subset I$ for all $D_i \in T$.

THEOREM 7. *If infinitely many principal prime ideals (f) are T-differential then every maximal ideal of R contains a nonzero T-differential principal prime ideal.*

PROOF. Let k_1 be the finitely generated extension of $\mathbf{Q}$ formed by adjoining the coefficients of all the q_{ij}. Following the proof of Theorem 6, we see that $R_1 \overset{\text{def}}{=} k_1[X]$ contains infinitely many distinct T-differential prime ideals (f_α). Suppose $D_i f_\alpha = h_{i\alpha} f_\alpha$. By Theorem 5, for each i the $\{h_{i\alpha}\}$ span a finite-dimensional vector space V_i over $\mathbf{Q}$. The direct sum $V_1 \oplus \cdots \oplus V_l$ is finite-dimensional, so there are positive integers $m_1, \ldots, m_r$, $n_1, \ldots, n_s$ $(r > 0)$, and $f_{\alpha_1}, \ldots, f_{\alpha_{r+s}} \in R_1$ such that for each $i = 1, \ldots, l$,

$$m_1 h_{i\alpha_1} + \cdots + m_r h_{i\alpha_r} - n_1 h_{i\alpha_{r+1}} - \cdots - n_s h_{i\alpha_{r+s}} = 0.$$

Construct F, G as in Theorem 2. For each i, $1 \leqslant i \leqslant l$, there are h_i satisfying $D_i F = h_i F$, $D_i G = h_i G$. Now use the construction of Theorem 1. $\square$

Let $m \subset R$ be a maximal ideal, let $k_1 = R/m$, and denote by $I \subset m$ the maximal T-differential ideal contained in m. I exists and is prime by [**S**, p. 33]; let $r = \operatorname{ht} I$ be the height of I. Obtain vectors $v_i = (\bar{q}_{i1}, \ldots, \bar{q}_{in}) \in k_1^n$ by reducing the polynomials appearing in the D_i modulo m.

THEOREM 8. *If $\{v_1, \ldots, v_l\}$ is linearly independent over k_1 then* $\operatorname{ht} I \leqslant n - l$.

PROOF. By [**S**, p. 34, Theorem 6] the local ring $(R/I)_{m/I}$ is regular, so we can find r elements $g_1, \ldots, g_r \in I$ such that the rank of the Jacobian (g_{iX_j}) modulo m is r. Let $w_j = (\bar{g}_{jX_1}, \ldots, \bar{g}_{jX_n})$. We have $v_i \cdot w_j = 0$ for $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant r$, so $r \leqslant n - l$. $\square$

It follows that in the case $l = n - 1$ there is a uniqueness theorem:

COROLLARY. *Suppose that $\{v_1, \dots, v_{n-1}\}$ is linearly independent over k_1. Then there can exist at most one height 1 T-differential prime $(f) \subset m$; for such (f), the local ring $(R/(f))_{m/(f)}$ is regular.* $\square$

4. Examples. To illustrate the preceding results, we consider derivations $D = p\partial/\partial X + q\partial/\partial Y$ of $R = k[X, Y]$ $(p, q \in R)$. Let $f \in R$ be an irreducible nonzero polynomial. On the curve $f = 0$, we have $f_X dx + f_Y dy = 0$, so the condition $pf_X + qf_Y = 0 \pmod f$ is equivalent to $-qdx + pdy = 0$. In other words, irreducible f satisfying $Df \in (f)$ correspond to integral curves $f = 0$ of the differential equation $pdy = qdx$. By Theorem 6, if there are infinitely many (algebraic) integral curves then each point $(a, b) \in k^2$ is contained in an integral curve. By Theorem 8, if $p(a, b) \neq 0$ or $q(a, b) \neq 0$ then at most one integral curve passes through (a, b).

As a first example, let $D = \partial/\partial X + Y\partial/\partial Y$. (Y) is the only nonzero principal prime differential ideal, for if $Df = hf$ and we write $f = f_1(X) + Yf_2(X, Y)$ we see that $\deg h = 0$ and so $f_1 = 0$.

Next, let $m \geqslant 1$ be an integer, and set

$$D = 2XY\frac{\partial}{\partial X} + \left(mY^2 - (m-1)X\right)\frac{\partial}{\partial Y}.$$

The solutions X, $Y^2 - X - cX^m$ $(c \in k)$ cover the plane. The degree of the solutions can be made as large as desired by increasing m. This illustrates the case of infinitely many solutions: for each k-rational point at least one solution is a factor of $G - cF$ $(c \in k)$ as in Theorems 1 and 6 (proof) and therefore has bounded degree, but it is not obvious how to determine the bound by inspection of D. This classical problem was studied, for example, by Poincaré [**P**]; he obtained partial results depending on the local behavior of solutions at the points where $p(a, b) = q(a, b) = 0$.

Return again to the case of several indeterminates and several derivations $T = \{D_1, \dots, D_{n-1}\}$ where now $l = n - 1$. Define a homogeneous 1-form

$$w_T = \det \begin{bmatrix} dX_0 & & \cdots & dX_n \\ X_0 & X_1 & \cdots & X_n \\ 0 & Q_{11} & \cdots & Q_{1n} \\ 0 & Q_{21} & \cdots & Q_{2n} \\ & & \cdots & \\ 0 & Q_{n-1,1} & \cdots & Q_{n-1,n} \end{bmatrix}$$

where the Q_{ij} are the q_{ij} made homogeneous of degree $r = \max_{i,j}(\deg q_{ij})$ by introducing a new indeterminate X_0; $Q_{ij}(X_0, \dots, X_n) = q_{ij}(x_1, \dots, x_n)X_0^r$, where $x_i = X_i/X_0$. Similarly, for $f \in R$, set $F(X_0, \dots, X_n) = f(x_1, \dots, x_n)X_0^d$, $d = \deg f$. We see that w_T is a Pfaffian 1-form and for T-differential f, we have (after some computation) that F is an algebraic solution of $w_T = 0$ (i.e. F divides $w_T \wedge dF$) in the sense of [**J**, pp. 81, 99; see also p. 102, Theorem 3.3]. For $n = 2$, the converse is true: if F divides $w_T \wedge dF$ then f is $T\,(= \{D_1\})$-differential.

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