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FACTORIZATION IN CODIMENSION ONE IDEALS OF GROUP ALGEBRAS

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ABSTRACT. It is shown that if G is a locally compact group and I is a closed, two-sided ideal with codimension one in $L^1(G)$, then $I^2 = I$.

Let G denote a locally compact group, $L^1(G)$ be its group algebra with respect to left invariant Haar measure and $M(G)$ be the algebra of bounded Borel measures on G. As usual, we will identify $L^1(G)$ with the ideal of $M(G)$ consisting of measures which are absolutely continuous with respect to Haar measure. For each Banach algebra A, define

$$A^2 = \left\{ \sum_{k=1}^{n} a_k b_k \mid a_k, b_k \in A, k = 1,\ldots,n; n \in \mathbf{N} \right\}.$$

A question asked by B. E. Johnson in connection with certain automatic continuity problems (see [**3**, Example 6.3]) is whether $I^2 = I$ when I is a closed, two-sided, codimension one ideal in $L^1(G)$. This question may be answered immediately when G is amenable, because in that case every codimension one ideal, I, in $L^1(G)$ has bounded approximate units (see [**5**]) and so, by Cohen's factorization theorem [**1**, 11.10], every element of I is a product of two others. Here we answer Johnson's question in the nonamenable case.

THEOREM. *Let G be a locally compact group and I be a closed two-sided ideal with codimension one in $L^1(G)$. Then $I^2 = I$.*

PROOF. Let χ_I be the continuous character on G such that $I = \{f \in L^1(G) \mid \int_G \chi_I(x) f(x)\,dx = 0\}$ (see [**4**, 23.7]). Then the operator T_I, defined by $T_I(f) = \chi_I f$, is an automorphism of $L^1(G)$ and $T_I(I) = \{f \in L^1(G) \mid \int_G f(x)\,dx = 0\} = I_0(G)$. Thus it suffices to prove the theorem in the case when $I = I_0(G)$. (This reduction was pointed out to me by B. E. Johnson.) Also, define $J_0(G) = \{\mu \in M(G) \mid \mu(G) = 0\}$.

Let f be in $I_0(G)$. Then, since $L^1(G)$ has bounded approximate units [**4**, 20.27], it follows, by Cohen's factorization theorem, that there are elements a and b in $L^1(G)$ and h in $I_0(G)$ such that $f = a * h * b$. We will show that

$$h = \sum_{k=1}^{4} \mu_k * v_k, \qquad (\mu_k, v_k \in J_0(G), k = 1,2,3,4).$$

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The theorem will then follow because

$$f = a * h * b = \sum_{k=1}^{4} (a * \mu_k) * (v_k * b),$$

where $a * \mu_k$ and $v_k * b$ are in $I_0(G)$ for $k = 1, 2, 3, 4$.

Now let r_1 and r_2 be the real and imaginary parts of h, so that r_1 and r_2 are real valued functions on G and $h = r_1 + ir_2$. Then, since h is in $I_0(G)$,

$$\int_G r_1(x)\,dx = 0 = \int_G r_2(x)\,dx,$$

and r_1 and r_2 are also in $I_0(G)$. Define functions r_j^+ and r_j^- for $j = 1, 2$ by

$$r_j^+(x) = \begin{cases} r_j(x), & \text{if } r_j(x) \geqslant 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$r_j^-(x) = \begin{cases} -r_j(x), & \text{if } r_j(x) \leqslant 0, \\ 0, & \text{otherwise,} \end{cases}$$

and define

$$\lambda_j = \int_G r_j^+(x)\,dx \quad \left(= \int_G r_j^-(x)\,dx \text{ because } r_j = r_j^+ - r_j^- \right).$$

Finally, if $r_j \neq 0$ (so that $\lambda_j > 0$), put $t_j^{\pm} = r_j^{\pm}/\lambda_j$.

With $t_j^{\pm}$ defined in this way we have that $t_j^{\pm}(x) \geqslant 0$ for every x in G and $\int_G t_j^{\pm}(x)\,dx = 1$, for $j = 1, 2$. Hence,

$$h = \lambda_1(e - t_1^-) - \lambda_1(e - t_1^+) + i\lambda_2(e - t_2^-) - i\lambda_2(e - t_2^+),$$

where $\|t_j^{\pm}\| = 1$ and e is the identity element in $M(G)$, so that $e - t_j^{\pm}$ is in $J_0(G)$ for $j = 1, 2$.

The Theorem now follows because, if t in $L^1(G)$ is such that $t(x) \geqslant 0$ for every x in G and $\|t\| = 1$, then $e - t$ has a square root in $J_0(G)$ defined by the binomial expansion

$$(e - t)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-t)^n.$$

The series converges because the coefficients $\{\binom{1/2}{n}\}_{n=0}^{\infty}$ form an l^1-sequence and $\|t^n\| = 1$ for every n. That $(e - t)^{1/2}$ is in $J_0(G)$ follows because $e - t$ is and because $J_0(G)$ is the kernel of a multiplicative linear functional on $M(G)$.

The last part of the proof in fact shows the following result for $J_0(G)$.

COROLLARY. *Let G be a locally compact group. Then $J_0(G)^2 = J_0(G)$.*

However, in spite of this, the theorem does not hold if $L^1(G)$ is replaced by $M(G)$. It is shown in **[2]** that, if G is a nondiscrete abelian group, then there is a codimension one, closed ideal I in $M(G)$ such that $I^2 \neq I$.

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