FACTORIZATION IN CODIMENSION ONE IDEALS
OF GROUP ALGEBRAS

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Abstract. It is shown that if $G$ is a locally compact group and $I$ is a closed, two-sided ideal with codimension one in $L^1(G)$, then $I^2 = I$.

Let $G$ denote a locally compact group, $L^1(G)$ be its group algebra with respect to left invariant Haar measure and $M(G)$ be the algebra of bounded Borel measures on $G$. As usual, we will identify $L^1(G)$ with the ideal of $M(G)$ consisting of measures which are absolutely continuous with respect to Haar measure. For each Banach algebra $A$, define

$$A^2 = \left\{ \sum_{k=1}^n a_k b_k \mid a_k, b_k \in A, k = 1, \ldots, n \right\}.$$

A question asked by B. E. Johnson in connection with certain automatic continuity problems (see [3, Example 6.3]) is whether $I^2 = I$ when $I$ is a closed, two-sided, codimension one ideal in $L^1(G)$. This question may be answered immediately when $G$ is amenable, because in that case every codimension one ideal, $I$, in $L^1(G)$ has bounded approximate units (see [5]) and so, by Cohen's factorization theorem [1, 11.10], every element of $I$ is a product of two others. Here we answer Johnson's question in the nonamenable case.

**Theorem.** Let $G$ be a locally compact group and $I$ be a closed two-sided ideal with codimension one in $L^1(G)$. Then $I^2 = I$.

**Proof.** Let $\chi_I$ be the continuous character on $G$ such that $I = \{ f \in L^1(G) \mid \int_G \chi_I(x)f(x) \, dx = 0 \}$ (see [4, 23.7]). Then the operator $T_f$, defined by $T_f(f) = \chi_I f$, is an automorphism of $L^1(G)$ and $T_I(I) = \{ f \in L^1(G) \mid \int_G f(x) \, dx = 0 \} = I_0(G)$. Thus it suffices to prove the theorem in the case when $I = I_0(G)$. (This reduction was pointed out to me by B. E. Johnson.) Also, define $J_0(G) = \{ \mu \in M(G) \mid \mu(G) = 0 \}$.

Let $f$ be in $I_0(G)$. Then, since $L^1(G)$ has bounded approximate units [4, 20.27], it follows, by Cohen's factorization theorem, that there are elements $a$ and $b$ in $L^1(G)$ and $h$ in $I_0(G)$ such that $f = a \ast h \ast b$. We will show that

$$h = \sum_{k=1}^4 \mu_k \ast v_k, \quad (\mu_k, v_k \in J_0(G), k = 1, 2, 3, 4).$$
The theorem will then follow because

\[ f = a \ast h \ast b = \sum_{k=1}^{4} (a \ast \mu_k) \ast (v_k \ast b), \]

where \( a \ast \mu_k \) and \( v_k \ast b \) are in \( I_0(G) \) for \( k = 1, 2, 3, 4. \)

Now let \( r_1 \) and \( r_2 \) be the real and imaginary parts of \( h \), so that \( r_1 \) and \( r_2 \) are real valued functions on \( G \) and \( h = r_1 + ir_2 \). Then, since \( h \) is in \( I_0(G) \),

\[ \int_G r_1(x) \, dx = 0 = \int_G r_2(x) \, dx, \]

and \( r_1 \) and \( r_2 \) are also in \( I_0(G) \). Define functions \( r_j^+ \) and \( r_j^- \) for \( j = 1, 2 \) by

\[ r_j^+(x) = \begin{cases} r_j(x), & \text{if } r_j(x) \geq 0, \\ 0, & \text{otherwise,} \end{cases} \]

\[ r_j^-(x) = \begin{cases} -r_j(x), & \text{if } r_j(x) \leq 0, \\ 0, & \text{otherwise,} \end{cases} \]

and define

\[ \lambda_j = \int_G r_j^+(x) \, dx = \int_G r_j^-(x) \, dx \text{ because } r_j = r_j^+ - r_j^- \].

Finally, if \( r_j \neq 0 \) (so that \( \lambda_j > 0 \)), put \( t_j^\pm = r_j^\pm / \lambda_j \).

With \( t_j^\pm \) defined in this way we have that \( t_j^\pm(x) \geq 0 \) for every \( x \) in \( G \) and \( \int_G t_j^\pm(x) \, dx = 1 \), for \( j = 1, 2 \). Hence,

\[ h = \lambda_1(e - t_1^-) - \lambda_1(e - t_1^+) + i\lambda_2(e - t_2^-) - i\lambda_2(e - t_2^+), \]

where \( \|t_j^\pm\| = 1 \) and \( e \) is the identity element in \( M(G) \), so that \( e - t_j^\pm \) is in \( J_0(G) \) for \( j = 1, 2 \).

The Theorem now follows because, if \( t \) in \( L^1(G) \) is such that \( t(x) \geq 0 \) for every \( x \) in \( G \) and \( \|t\| = 1 \), then \( e - t \) has a square root in \( J_0(G) \) defined by the binomial expansion

\[ (e - t)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-t)^n. \]

The series converges because the coefficients \( \{\binom{1/2}{n}\}_{n=0}^{\infty} \) form an \( l^1 \)-sequence and \( \|t^n\| = 1 \) for every \( n \). That \( (e - t)^{1/2} \) is in \( J_0(G) \) follows because \( e - t \) is and because \( J_0(G) \) is the kernel of a multiplicative linear functional on \( M(G) \).

The last part of the proof in fact shows the following result for \( J_0(G) \).

**Corollary.** Let \( G \) be a locally compact group. Then \( J_0(G)^2 = J_0(G) \).

However, in spite of this, the theorem does not hold if \( L^1(G) \) is replaced by \( M(G) \).

It is shown in [2] that, if \( G \) is a nondiscrete abelian group, then there is a codimension one, closed ideal \( I \) in \( M(G) \) such that \( I^2 \neq I \).

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References


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