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A COUNTEREXAMPLE TO A CONJECTURE IN LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of the present paper is to give a simple counterexample to a conjecture [3] in linear second-order differential equations.

1. Introduction. Consider the following differential equation

$$u'' + a(t) \cdot u = 0 \tag{1}$$

where $a(t)$ is a nondecreasing, positive and unbounded function in $C'[T, \infty)$. It is well known that the hypotheses on $a(t)$ do not imply that every solution of (1) satisfies the condition

$$u(t) \to 0 \quad \text{as}\, t \to \infty. \tag{2}$$

L. A. Gusarov [2] has shown that under the additional hypothesis that $a'(t)$ is of bounded variation on $[T, \infty)$, the solutions of (1) satisfy condition (2). Under these assumptions, $a'(t)$ has a finite, nonnegative limit as $t \to \infty$. A. Meir, D. Willett and J. S. W. Wong [3] have proved the following theorem.

THEOREM 1. *If there exists a positive function $p(t) \in C'[0, \infty)$ such that*

$$\int_0^\infty \frac{dt}{p(t)} = +\infty, \quad \liminf_{t\to\infty} \frac{p'(t)}{p(t) \cdot a^{1/2}(t)} \geq 0 \quad \textit{and} \quad \liminf_{t\to\infty} \frac{a'(t) \cdot p(t)}{a(t)} > 0,$$

then the solutions of (1) *satisfy condition* (2).

From this theorem it follows that if $a'(t)$ is ultimately bounded and bounded away from zero, then all solutions of (1) satisfy (2). The following question presents itself: does the condition that $a'(t) \to 0$ as $t \to \infty$ (or that $\limsup a'(t) < \infty$) imply that condition (2) holds for all solutions of (1)?

Meir, Willett and Wong [3] conjectured that if in Theorem 1 the last condition is replaced by the condition

$$\lim_{t\to\infty} a'(t) \cdot p(t)/a(t) = 0,$$

then the conclusion remains valid. If this conjecture were true, we could answer our question in the affirmative (simply set $p(t) \equiv 1$). However, the following theorem shows that the conjecture is false.

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2. The counterexample. We can summarize our results as follows:

THEOREM 2. *For each* $\alpha > 0$, *there exists a positive function* $a(t) \in C^\infty[0,\infty)$ *such that* $a(t) \to \infty$, $a'(t) \geq 0$, $a'(t) = o(\log^{-\alpha} t)$ *and such that at least one solution* $u(t)$ *of* (1) *satisfies the condition* $\limsup_{t\to\infty} |u(t)| > 0$.

Without loss of generality, we replace the condition $a'(t) = o(\log^{-\alpha} t)$ by $a'(t) = O(\log^{-l} t)$, where l is an integer ($l > \alpha$).

The proof of Theorem 2 is based on a method used by A. S. Galbraith, E. J. McShane and G. B. Parrish [1], and D. Willett [5]. The following lemma, which was established by Willett [5], will be used in the proof of Theorem 2.

LEMMA. *Let* $u(t)$ *be a solution of* (1), *and let* μ *be a positive number such that* $a(t) \geq \mu^2$ *for all* $t \in [0,\infty)$. *Then* $u'(t)$ *has at least one zero in each interval of length* $2\pi/\mu$.

PROOF OF THEOREM 2. Consider the functions $A(t)$ and $B(t)$ defined by

$$A(t) = \begin{cases} \exp(1 - 1/t^2) & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases} \qquad B(t) = A[1 - A(1-t)].$$

Clearly, $B(t)$ is a nondecreasing C^∞-function with values in [0,1], and it satisfies the conditions $B(t) = 0$, $t \leq 0$, and $B(t) = 1$, $t \geq 1$.

Let $t_1 = 0$, $q_1 = 1/2$, $\beta_0 = 16\pi^2$ and $\beta_1 = \beta_0 + 1$. Define $a_1(t)$ by condition $a_1(t) = \beta_0 + (\beta_1 - \beta_0) \cdot B(2t)$. Let u_1 denote the unique solution of the initial value problem

$$u_1'' + a_1(t) \cdot u_1 = 0, \quad u_1(0) = 1 \quad \text{and} \quad u_1'(0) = 0.$$

By the Lemma, there exists a point t_2 ($1/2 \leq t_2 < 1$) such that $u_1'(t_2) = 0$.

The following construction is inductive. We choose a sequence $\{\beta_n\}$, a sequence $0 = t_1 < q_1 < t_2 < q_2 < \cdots$, and a set of functions $u_n(t)$ ($n = 1, 2, \ldots$) such that

$$q_n - t_n = \begin{cases} \min[n^{-1} \cdot \log^l n], & \text{if } n \geq 2, \\ 1/2, & \text{if } n = 1, \end{cases}$$

$$n - 3/2 \leq t_n \leq n - 1,$$

$$\beta_n = \frac{1}{n} + \beta_{n-1} = \sum_{k=1}^{n} \frac{1}{k} + 16\pi^2,$$

$$a_n(t) = \beta_{n-1} + (\beta_n - \beta_{n-1}) \cdot B((t - t_n)/(q_n - t_n)),$$

$$u_n'' + a_n(t) \cdot u_n = 0, \qquad u_n(t_n) = u_{n-1}(t_n), \qquad u_n'(t_n) = u_{n-1}'(t_n) = 0.$$

Letting $\chi[t_n, t_{n+1})$ denote the characteristic function of the half-open interval, we set

$$a(t) = \sum_{n=1}^{\infty} a_n(t) \cdot \chi[t_n, t_{n+1}) \quad \text{and} \quad u(t) = \sum_{n=1}^{\infty} u_n(t) \cdot \chi[t_n, t_{n+1}).$$

We see that $a(t)$ is a positive, nondecreasing function belonging to $C^\infty[0,\infty)$, and that $u(t)$ satisfies the differential equation (1). Since $\beta_n \to \infty$ as $n \to \infty$, it follows that $a(t) \to \infty$ as $t \to \infty$.

We now establish a bound on $a'(t)$. Differentiating $a(t)$, we obtain the equation

$$a'(t) = \begin{cases} \dfrac{1}{q_n - t_n} \cdot B'\left(\dfrac{t - t_n}{q_n - t_n}\right) \cdot (\beta_n - \beta_{n-1}) & \text{for } t_n \leq t \leq q_n, \\ 0 & \text{for } q_n \leq t \leq t_{n+1}. \end{cases}$$

Since $B(t)$ is a C^∞-function having compact support, $B'(t)$ is bounded by some positive number M. Hence, for $t_n \leq t \leq t_{n+1}$ and $n \geq 2$:

$$a'(t) \leq \frac{1}{q_n - t_n} \cdot (\beta_n - \beta_{n-1}) \cdot M < 2 \cdot M \cdot \log^{-l} n. \tag{3}$$

For $t_n \leq t \leq t_{n+1}$ and $n \geq 5$, it follows from the condition $n - 3/2 \leq t_n \leq n - 1$ that $\log n \geq \log t_{n+1} \geq \log t \geq 1$. Combining this with (3), we obtain the estimate $a'(t) = O(\log^{-l} t)$.

To show that $\limsup_{t\to\infty} |u(t)| > 0$, we choose numbers

$$\varphi_n = 1/2 \cdot (q_n - t_n)^2 \cdot a(q_n) \qquad (n \geq 2).$$

Since $\lim_{t\to\infty} (\log^k t) \cdot t^{-1/2} = 0$ for each positive integer k, there exists an integer N such that

$$16\pi^2 (\log^{2l} n) \cdot n^{-1/2} \leq 1 \quad \text{and} \quad 2 \cdot (\log^{2l+1} n) \cdot n^{-1/2} \leq 1,$$

whenever $n \geq N$. Since each φ_n $(n \geq N)$ satisfies the inequalities

$$\begin{aligned} \varphi_n &= \frac{1}{2}\left[\sum_{k=1}^{n} \frac{1}{k} + 16\pi^2\right] \cdot n^{-2} \cdot \log^{2l} n \leq \frac{1}{2}\left[n^{-3/2} + (1 + \log n) n^{-2} \cdot \log^{2l} n\right] \\ &\leq 2^{-1} \cdot [n^{-3/2} + 2 \cdot n^{-2} \cdot \log^{2l+1} n] \leq n^{-3/2} \end{aligned}$$

we see that $\sum_{n=1}^\infty \varphi_n \leq \sum_{n=1}^\infty (1/n^{3/2}) < \infty$.

We now show that

$$|u(t_n)| \cdot [1 - \varphi_n] \leq |u(t_{n+1})| \tag{4}$$

for each of the points t_n. By Taylor's theorem, we set

$$t_n \leq c \leq q_n; \qquad u(q_n) = (q_n - t_n)^2 \cdot u''(c)/2! + u(t_n).$$

We note that $|u''(c)| = a(c) \cdot |u(c)|$ and that $a(c) \leq a(q_n)$. It is well known [4, Part 2, p. 28] that the values $|u(\eta_i)|$ determined by the points η_i $(i = 1, 2, \dots)$ where $u'(\eta_i) = 0$ form a decreasing sequence. Therefore, $|u(c)| \leq |u(t_n)|$. From these observations we obtain the relations

$$\begin{aligned} |u(q_n)| &= |u(t_n) + (q_n - t_n)^2 \cdot u''(c)/2!| \\ &\geq \left[1 - \tfrac{1}{2}(q_n - t_n)^2 \cdot a(q_n)\right] \cdot |u(t_n)| = [1 - \varphi_n] \cdot |u(t_n)|. \end{aligned} \tag{5}$$

To estimate $|u(t_{n+1})|$, we integrate the expression

$$u'' \cdot u' + a \cdot u' \cdot u = 0$$

by parts and obtain the equation

$$a(t_{n+1}) \cdot u^2(t_{n+1}) = (u'(q_n))^2 + a(q_n) \cdot u^2(q_n) + \int_{q_n}^{t_{n+1}} u^2(t) \cdot a'(t)\, dt.$$

From this we deduce that

$$u^2(t_{n+1}) \geq u^2(q_n) \cdot a(q_n) \cdot a^{-1}(t_{n+1}) = u^2(q_n).$$

Combining (5) with this inequality, we obtain (4).

Since $\sum_{n=1}^{\infty} \varphi_n < \infty$, there exists a positive integer N such that $0 < \varphi_n < 1$ for $n \geq N$. From inequality (4), we see that

$$|u(t_N)| \cdot \prod_{k=N}^{n} (1 - \varphi_k) \leq |u(t_{n+1})|.$$

Since the product $\prod_{k=N}^{\infty}(1 - \varphi_k)$ converges to some positive number, we deduce that $\limsup_{t\to\infty} |u(t)| > 0$, and this completes the proof.

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