UNIQUENESS OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WHEN THE INITIAL SURFACE IS CHARACTERISTIC AT A POINT

LETITIA J. KORBLY

ABSTRACT. Uniqueness in the Cauchy problem for hyperbolic operators degenerate at a point on the initial surface depends on the values of the coefficients of the lower order terms. If the operator \( P \) is doubly characteristic at the origin with respect to the \( t = 0 \) line, \( P \) has uniqueness for functions which are smooth enough if the coefficient of the \( D_t \) term does not lie in a certain discrete set of numbers.

Introduction. We consider operators of the form

\[
P = A(x, t)D_t^2 + 2B(x, t)D_xD_t - C(x, t)D_x^2 + \lambda(x, t)D_t + R_1
\]

where all the coefficients are real \( C^\infty \) functions, \( A \) is nonnegative, \( C \) is positive, \( R_1 \) is a first order linear operator in \( D_x \),

\[
D_x = \frac{\partial}{\partial x} \quad \text{and} \quad D_t = \frac{\partial}{\partial t}.
\]

When \( P \) is doubly characteristic at the origin with respect to the surface given by \( t = 0 \), uniqueness of the solution to the problem:

\[
Pu = F, \quad (x, t) \in \Omega, \text{ a neighborhood of the origin, } t \geq 0,
\]

\[
u = u_0(x), \quad \text{if } t = 0,
\]

\[
D_tu = u_1(x), \quad \text{if } t = 0,
\]

is shown to depend on the values of the lower order terms, principally \( \lambda \). If \( \lambda \) lies outside a certain discrete set of values the solutions to the problem will be unique whenever \( \mu \) is smooth enough.

If \( \lambda \) lies in this discrete set of values, uniqueness will depend on \( R_1 \). The dependence on the lower order terms is very complex. There is always uniqueness when

\[
\lambda_0 = \lambda - D_xB - D_tA/2 \geq 0.
\]

As \( \lambda_0 \) becomes more negative, \( u \) must be assumed to be smoother to get the same result.

There are some values of the coefficients for which no uniqueness results can be obtained. If \( c > 0 \),

\[
Pd^2 u = x^2 D_t^2 u - c^2 D_x^2 u - (2n + 1)c D_t u
\]

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and

\[ \tilde{X}u = xD_t u - cD_x u, \]

then

\[ P \tilde{X}^N h(ct - x^2/2) = 0, \quad h \in C_c^\infty(R). \]

The assertion is proved by taking the support of \( h \) to be the positive halfline. Finally, if \( P \) is as in (3), \( P + \gamma \) is shown to have uniqueness if \( \gamma \) is a nonzero function.

**The operators.** \( P \) is characteristic with respect to the initial surface at the origin when \( A \) vanishes at the origin, and doubly characteristic with respect to this surface when both \( B \) and \( \text{grad} \ A \) also vanish there.

In order to use the concatenation technique of Višik and Grušin, [12], it is necessary to factor the principal part of the operator in a neighborhood of the origin which we will take to be \( \Omega \). The following hypotheses are assumed:

(Ha) There is a \( C^\infty \) function, \( a \), such that

\[ A = a^2 \quad \text{and} \quad D_x a = a_x > 0. \]

(Hb) There is a \( C^\infty \) function, \( b \), such that

\[ B = ba. \]

Then

\[ P = a^2 D_t^2 + 2ab D_t D_x + b^2 D_x^2 - (C + b^2) D_x^2 + \lambda D_t + R_1. \]

Let

\[ c^2 = C + b^2, \quad \text{where} \ c > 0. \]

Define

\[ \tilde{Y} = aD_t + bD_x + cD_x \quad \text{and} \quad \tilde{X} = aD_t + bD_x - cD_x. \]

So

\[ P = \tilde{Y} \tilde{X} + \lambda' D_t + R_2, \]

where \( R_2 \) is a first order operator in \( D_x \) and

\[ \lambda' = \lambda - (b + c)a_x. \]

This form is used in the concatenations. The commutator, \([\tilde{Y}, \tilde{X}]\), of \( \tilde{Y} \) and \( \tilde{X} \) acting on a function is

\[ [\tilde{Y}, \tilde{X}]u = (\tilde{Y} \tilde{X} - \tilde{X} \tilde{Y})u = (2a_x cD_t + R_3)u, \]

where \( R_3 \) is a first order operator in \( D_x \). The hypotheses (1) and (Ha) guarantee \( a_x c > 0 \).

**DEFINITION 1.** The following norms are used:

\[ ||u||_2^2 = ||u||_0^2 = \iint_{\Omega} |u|^2 \, dx \, dt, \quad u \in \mathcal{H}^2(\Omega). \]

Then

\[ ||u||_{N+1}^2 = ||D_t u||_N^2 + \sigma^2 ||u||_N^2 + ||D_x u||_N^2. \]

The following norm is taken from [1]:

\[ ||u||_{1,1}^2 = ||aD_t u||_0^2 + \sigma^2 ||au||_0^2 + ||D_x u||_0^2. \]
Note that for all \( u \) in \( C^\infty_c(\Omega) \), and \( \sigma > \sigma_0 \):
\[
\|u\|_{1,1} \leq \|u\|_1.
\]
Also, if \( u \in C^\infty_c(\Omega) \),
\[
\|(D_t + \sigma)u\|^2 = \|D_t u\|^2 + \sigma^2 \|u\|^2.
\]
Similar statements are true when other norms are used.

**Definition 2.** An operator, \( P \), has \((UCP)_N\) in an open set, \( \Omega \), if, whenever \( u \) is an element of \( H^N(\Omega) \), and \( \text{supp} \ u \cap \{ t \leq 0 \} \) is compact, then \( Pu = 0 \) in \( \Omega \), implies \( u = 0 \) in a neighborhood of \( \text{supp} \ u \cap \{ t \leq 0 \} \).

Let \( \Omega^- = \Omega \cap \{ t \leq 0 \} \). The following lemma is only stated because its proof is an easy modification of that in [8].

**Lemma 1.** If there are constants \( c \) and \( \sigma_0 \) such that for every function \( v \) in \( C^\infty_c(\Omega) \), and every \( \sigma > \sigma_0 \), the Carleman estimate
\[
C \|e^{-\sigma t}Pv\|_N \geq \|e^{-\sigma t}v\|_0
\]
holds, then \( P \) has \((UCP)_{N+m}\). Here \( m \) is the order of \( P \).

Define
\[
P_\sigma u = \exp(-\sigma t)P(\exp(\sigma t) u).
\]
The substitution
\[
v = \exp(\sigma t) u
\]
shows that (6) is equivalent to the following: there is a constant \( c \), such that
\[
c \|P_\sigma u\|_N \geq \|u\|_0, \quad u \in C^\infty_c(\Omega), \quad \sigma > \sigma_0.
\]
This is the estimate used in proving the theorem.

**Proof of the theorem when \( \lambda_0 \) is positive.** Let \( A_t \) be the partial of \( A \) with respect to \( t \), \( B_x \), the partial of \( B \) with respect to \( x \), and \( C_t \) and \( C_x \) be the partials of \( C \) with respect to \( t \) and \( x \) respectively.

**Lemma 2.** If \( P \) is as in (1) where \( A \) is nonnegative and \( C \) is positive, and
\[
0 < L < \lambda_0 = \lambda(0,0) - B_x(0,0) - A_t(0,0)/2,
\]
then there are \( c > 0 \) and \( \sigma_0 \), depending only on \( L \) and the coefficients of \( P \) such that if \( \Omega \) is a sufficiently small neighborhood of the origin,
\[
\langle P_\sigma u, (D_t + \sigma)u \rangle \geq \sigma \|u\|_{1,1}^2 + L \|u\|_1^2, \quad u \in C^\infty_c(\Omega), \quad \sigma > \sigma_0.
\]

**Proof.** Integration by parts in \( D_t \) shows
\[
\langle A(D_t + \sigma)^2u, (D_t + \sigma)u \rangle = \sigma \|A^{1/2}(D_t + \sigma)u\|^2 - \langle A_t/2(D_t + \sigma)u, (D_t + \sigma)u \rangle
\]
whenever \( u \) is in \( C^\infty_c(\Omega) \).

Integration by parts in \( D_x \) shows that if \( u \in C^\infty_c(\Omega) \),
\[
\langle 2BD_x(D_t + \sigma)u, (D_t + \sigma)u \rangle = -(B_x(D_t + \sigma)u, (D_t + \sigma)u).
\]
An integration by parts in \( D_t \) followed by another in \( D_x \) shows if \( u \in C^\infty_c(\Omega) \),
\[
\langle CD_x^2u, (D_t + \sigma)u \rangle = \sigma \|C^{1/2}D_xu\|^2 - \langle (C_t/2)D_xu, (D_t + \sigma)u \rangle + \langle (C_x/2)D_xu, (D_t + \sigma)u \rangle.
\]
We use (1), (7), (8), and the last equation, (9), to get if \( u \in C_c^\infty(\Omega) \),
\[
\langle P_\phi u, (D_t + \sigma)u \rangle = \sigma \|a(D_t + \sigma)u\|^2 + \|\sigma C - C_x/2\|^2/2 \|D_x u\|^2
\]
\[
+ \|\langle \lambda_0 \rangle^{1/2}(D_t + \sigma)u\|^2 + \langle R_4 u, (D_t + \sigma)u \rangle
\]
where \( R_4 \) is a first order operator in \( D_x \).

Schwarz’s inequality implies that for some \( R \) and any positive \( \epsilon \),
\[
\langle R_4 u, (D_t + \sigma)u \rangle \leq \epsilon \|D_t + \sigma\|^2 + 1/(2\epsilon) \|Ru\|^2, \quad u \in C_c^\infty(\Omega).
\]
Fix \( \epsilon \) so that \( L = \lambda_0 - \epsilon > 0 \) after shrinking \( \Omega \) if necessary. Then choose \( \sigma_0 \) so that \( \sigma_0 C - C_x/2 - R/2\epsilon \) is positive. This implies that the first three terms on the right in (10) dominate the last term on the right, which proves the lemma. □

**Corollary.** If \( P \) is as in (1) and (Ha) and (Hb) hold, the conclusion of Lemma 2 is true if
\[
\lambda - a_x b > L > 0.
\]

**Proof.** Both \( ab_x \) and \( A_t/2 = aat \) vanish at the origin.

**Proof of the theorem when \( \lambda_0 \) is negative.** It is easy to show that \( P_\phi u \) dominates \( Y^N u \) if \( \lambda_0 \) is not less than \(-2Na_x c\), where
\[
Y u = e^{-\sigma t}Y(e^{\sigma t} u), \quad u \in C_c^\infty(\Omega),
\]
and
\[
Y^{N+1} u = YY^N u, \quad u \in C_c^\infty(\Omega).
\]
But no inequality which is uniform in \( \sigma \) bounds \( Yu \) away from 0 in a neighborhood of the origin. To prove the theorem when \( \lambda_0 < 0 \), more structure is necessary. The next lemma handles the commutators between \( Y^N \) and \( P_\phi \). Let
\[
X u = e^{-\sigma t}X(e^{\sigma t} u), \quad u \in C_c^\infty(\Omega).
\]

**Lemma 3.** If \( P \) is as in (1) and (Ha) and (Hb) are satisfied, then
\[
Y^N P_\phi u = (P_\phi + N[Y, X])Y^N u + \sum_{k < N} L_k Y^k u, \quad u \in C_c^\infty(\Omega).
\]
Here \( L_k \) is a first order operator in \( D_x \) and \( D_t \).

**Proof.** The proof is by induction. The lemma is clearly true when \( N = 0 \). The induction hypothesis is (11). Then if \( u \in C_c^\infty(\Omega) \),
\[
Y^{N+1} P_\phi u = Y \left( (XY + \lambda(D_t + \sigma) + R_2 + N[Y, X])Y^N u + \sum_{k < N} L_k Y^k u \right).
\]
\( Y \) is commuted with the rest of the right-hand side above. The commutator satisfies
\[
[Y, XY] = [Y, X]Y.
\]
All the other commutators are first order. This recovers the inductive hypothesis. The lemma follows. □

**Corollary.** If \( P \) is as in (1) and satisfies (Ha) and (Hb), and if
\[
\lambda(0,0) - a_x(0,0)b(0,0) = \lambda_0(0,0) > -2Na_x(0,0)c(0,0)
\]
and
\[
P^N = P_\phi + N[Y, X]
\]
then there is some neighborhood \( \Omega \) of the origin where \( P^N \) satisfies the hypotheses of Lemma 2.
Proof. Define

\[ \lambda_N = \lambda_0 + 2Na_xc \]

then the hypothesis of the corollary implies that in some neighborhood of the origin \( \lambda_N \) is strictly positive, and must be bounded away from 0 in some neighborhood of the origin which is taken to be \( \Omega \). But

\[ P_N = XY + \lambda_N D_t + R_N \]

where \( R_N \) is a first order operator in \( D_x \). This proves the corollary. \( \square \)

Now the main theorem will be proved.

Theorem 1. If \( P \) is as in (1) and (Ha) and (Hb) are satisfied then \( P \) has (UCP)\( N+2 \) if

(h1) \( \lambda' + 2ka_x(0,0)c(0,0) \neq 0 \) for any \( k = 1, 2, 3, \ldots \)

and

(h2) \( \lambda_0 + 2Na_x(0,0)c(0,0) > 0 \).

Proof. Lemma 3 implies that there are constants \( c, r, \sigma_0 \) such that if \( u \in C_c^\infty(\Omega) \) and \( \sigma > \sigma_0 \),

\[ c\|P_u\|_N \geq \|Y^NP_u\|_0 \geq \|P_N^N Y^N u\|_0 - S_0. \]

Here

\[ S_N = c\|D u\|_N \leq \|u\|_1. \]

The corollary to Lemma 3 and (h2) establish that \( P_N \) satisfies the hypothesis of Lemma 2, so there are constants \( c \) and \( \sigma_0 \), such that if \( u \in C_c^\infty(\Omega) \) and \( \sigma > \sigma_0 \),

\[ c\|P_N^N Y^N u, (D_t + \sigma)Y^N u\| \geq \|Y^N u\|_1^2 + \|Y^N u\|_1. \]

Since

\[ \| (D_t + \sigma)u \|_0 \leq \|u\|_1 \]

and because the 1,1 norm is smaller than the 1 norm, (14) implies there are constants, \( c \) and \( \sigma_0 \), such that if \( u \in C_c^\infty(\Omega), \sigma > \sigma_0 \) and \( N \geq 1, \)

\[ c\|P_u\|_N \geq \|Y^N u\|_1 + \|Y^N u\|_1. \]

Substitution of the last inequality into (13) implies there is a constant \( c \) such that if \( u \in C_c^\infty(\Omega) \) and \( \sigma > \sigma_0 \),

\[ c\|P_u\|_N \geq \|Y^N u\|_1 + \|Y^N u\|_1 - S_0. \]

This inequality is the first step in an inductive argument. The inductive hypothesis is that there are constants \( c, r, \sigma_0 \), depending on \( J, N, \) and \( P \), but not on \( u \in C_c^\infty(\Omega) \) or \( \sigma > \sigma_0 \), such that

\[ c\|P_u\|_N \geq \sigma^J + \|Y Y^N-J u\|_1 + \|X Y^N-J u\|_1 - S_J. \]

We now try to recover the inductive hypothesis for \( J+1 \). The inductive hypothesis for \( J \) implies there are constants \( c', r', \) and \( \sigma_0 \) such that if \( u \in C_c^\infty(\Omega) \) and \( \sigma > \sigma_0 \),

\[ c'\|P_u\|_N \geq \sigma^{J+1} + \|X Y Y^N-J-1 u\|_1 - S_J. \]
But the definition of the norms implies there is a constant $c''$ such that
\[ c''||P_\phi u||_N \geq \sigma^{J+1}||Y^{N-J-1}P_\phi u||_0, \quad u \in C_\infty^\sigma(\Omega), \quad \sigma > \sigma_0. \]

Lemma 3 applied to the right-hand side of the inequality above implies the existence of constants $c'''$, $r''$, and $\sigma_0$ such that if $u \in C_\infty^\sigma(\Omega)$ and $\sigma > \sigma_0$,
\[ (17) \quad c''||P_\phi u||_N \geq \sigma^{J+1}||P_\phi^{N-J-1}Y^{N-J-1}u||_0 - S_{J+1}. \]

Subtracting (16) from the last inequality gives the following for $u \in C_\infty^\sigma(\Omega)$ and $\sigma > \sigma_0$:
\[ \zeta||P_\phi u||_N \geq \sigma^{J+1}||\lambda_{N-J-1}(D_t + \sigma)Y^{N-J-1}u||_0 - \sigma^Jr'||Y^{N-J-1}u||_1 - S_{J+1}. \]

Here $r''$ and $\sigma_0$ have been increased to absorb all the terms of $S_J$ but the $(N - J - 1)$st. The $(N - J - 1)$st term is the one which must be dominated to recover the inductive hypothesis.

Since (h2) guarantees that $\lambda_{N-J-1} = 0$ in $\Omega$, we can add a multiple, $m$, of the last inequality to (17) to get for each $u \in C_\infty^\sigma(\Omega)$ and $\sigma > \sigma_0$,
\[ c''||P_\phi u||_N \geq \sigma^{J+1}||P_\phi^{N-J-1}u||_0 - \sigma^Jr'\lambda_{N-J-1}Y^{N-J-1}u||_1 - S_{N+1}, \]

where $c' = mc + c'''$. The multiple $m$ is chosen so that $P_\phi^{N-J-1}$ satisfies the hypothesis of Lemma 2. This now implies that for some new $c''$ and $\sigma_0$, if $u \in C_\infty^\sigma(\Omega)$ and $\sigma > \sigma_0$,
\[ c''||P_\phi u||_N \geq \sigma^{J+1}(\sigma||Y^{N-J-1}u||_{1,1} + ||Y^{N-J-1}u||_1) - \sigma^Jr'Y^{N-J-1}u||_1 - S_{N+1}. \]

If $\sigma_0$ is taken bigger than $r$, then the inductive hypothesis is recovered. This proves the estimate. The theorem follows from Lemma 1. □

With the introduction of more machinery, as in [1 and 2], a slightly stronger theorem can be proved. It is possible to show that the conditions of Theorem 1 imply $P$ has $(UCP)_{N+1}$. It is probably not possible to conclude that $P$ has uniqueness for functions with continuous second partials when $\lambda_0$ is negative. It is also possible to strengthen the conclusion of Theorem 2 slightly, but the gain is small compared to the increase in difficulty of the proof.

**COROLLARY.** If $P$ is as in (1) and (Ha) and (Hb) are satisfied, then $P$ has $(UCP)_{N+2}$ if
\[ (h1)' \quad \lambda \neq (b - (2k - 1)c)a_\omega, \quad k = 1, 2, 3, \ldots, \]
and
\[ (h2)' \quad \lambda > (b - 2Nc)a_\omega. \]

**PROOF.** The proof that (h2)' is equivalent to (h2) is exactly as in the corollary to Lemma 2. The equivalence of (h1)' to (h1) follows directly from (2), the definition of $\lambda_0$.

**COROLLARY 2.** If $P$ is as in (1) and (Ha) and (Hb) hold, then $P$ has $(UCP)_{N+2}$ if for every positive integer $k$, when $t = 0$,
\[ (h1)'' \quad \lambda(0, 0) \neq \lim_{x \to 0} \{A_\omega[B - (2k - 1)(AC + B^2)^{1/2}]/(2A)\} \]
and when $t = 0$,
\[ (h2)'' \quad \lambda(0, 0) > \lim_{x \to 0} \{A_\omega[B - 2N(AC + B^2)^{1/2}]/(2A)\}. \]
PROOF. This follows directly from the definitions of $a$, $b$, and $c$. We have
\[ a_x = A_x/(2A^{1/2}) \quad \text{and} \quad b_a = A_xB/(2A). \]
Also
\[ c_a = A_x(AC + B^2)^{1/2}/(2A). \]
The final theorem shows the complex dependence of uniqueness on the terms $R_N$ when $\lambda$ violates (h1).

**Theorem 2.** Let $\gamma$ be a nonvanishing function,
\[ P\gamma = (XY + N[Y,X] + \gamma)u, \]
\[ \gamma = 0 \quad \text{and} \quad [Y,[Y,X]] = 0. \]
Then $P$ has $(UCP)_{N+4}$.

**PROOF.** Application of Lemma 2 shows there are constants $c$ and $\sigma_0$, depending on $P$ but not on $u \in C^\infty_c(\Omega)$ or $\sigma > \sigma_0$, such that
\[ c\|P\phi u\|_{N+2} \geq \|Y^{N+1}P\phi u\|_1 = \|P^{N+1}Y^{N+1}u\|_1. \]
Here $P^{N+1}$ satisfies the hypothesis of Lemma 2. If for $u \in C^\infty_c(\Omega)$,
\[ \|u\|_{2,1}^2 = \|u_x\|_{2,1}^2 + \|u_x\|_{1,1}^2 + \sigma^2\|u\|_{1,1}^2, \]
an easy extension of Lemma 2 shows there are constants $c$ and $\sigma_0$ such that
\[ c\|P\phi u\|_{N+2} \geq \sigma\|Y^{N+1}u\|_{2,1} + \|Y^{N+1}u\|_2, \quad u \in C^\infty_c(\Omega), \sigma > \sigma_0. \]
This implies there are $c'$ and $\sigma_0$ such that if $u \in C^\infty_c(\Omega)$ and $\sigma > \sigma_0$,
\[ c'\|P\phi u\|_{N+2} \geq \sigma\|XY^{N}u\|_1. \]
But the definitions of the norms imply that for some $c$ and $\sigma_0$, if $u \in C^\infty_c(\Omega)$ and $\sigma > \sigma_0$
\[ (c' + c')\|P\phi u\|_{N+2} \geq \sigma\|Y^{N}P\phi u\|_1 = \sigma\|P^{N}Y^{N}u\|_1. \]
Here a whole derivative has been lost. Subtracting (11) from (18) implies that if $u \in C^\infty_c(\Omega)$ and $\sigma > \sigma_0$,
\[ (c'' + c')\|P\phi u\|_{N+2} \geq \sigma\|Y^{N}u\|_1. \]
The induction starts here. The inductive hypothesis is: there are $c$ and $\sigma_0$ such that if $u \in C^\infty_c(\Omega)$ and $\sigma > \sigma_0$,
\[ c\|P\phi u\|_{N+2} \geq \sigma^{N+1}\|Y^{N}u\|_1. \]
The induction is similar to that in Theorem 1. The hypotheses of this theorem guarantee all the commutators except $[Y,X]$ vanish. When $J = N$, an estimate which proves the theorem is obtained. \qed
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA, BIRMINGHAM, ALABAMA 35294