TAMING COMPACTA IN $E^4$

JOHN J. WALSH AND DAVID G. WRIGHT

ABSTRACT. A compactum $X$ in Euclidean 4-space $E^4$ is shown to be tame if its projection into $E^3$ is 1-dimensional and if $\dim L \cap X \leq 0$ for each vertical line $L$ in $E^4$. An essential ingredient is the result due to J. L. Bryant and D. L. Sumners that a 1-dimensional compactum in a 3-dimensional hyperplane of $E^4$ is tame in $E^4$.

A 0-dimensional or 1-dimensional compact subset $X$ of Euclidean 4-space $E^4$ is said to be tame in $E^4$ provided the embedding dimension of $X$ is $\leq 1$; i.e., for each open cover $\mathcal{U}$ of $E^4$ and closed 2-dimensional subpolyhedron $K \subset E^4$, there is a self-homeomorphism $h$ of $E^4$ $\mathcal{U}$-close to the identity with $h(K) \cap X = \emptyset$. For 0-dimensional compacta and 1-dimensional polyhedra, this notion of tameness is equivalent to the usual one $[Br_1, Br_2]$. Depending on the notion of tameness to be verified, either the Klee trick $[K]$ or the fact that 1-dimensional subpolyhedra of $E^3$ can be pushed off 0-dimensional compact subsets of $E^3$ establishes that a 0-dimensional compact subset of a 3-dimensional hyperplane of $E^4$ is tame in $E^4$. The examples of Bothe $[Bo]$ and McMillan and Row $[M-R]$ of tangled embeddings of 1-dimensional compacta in $E^3$ necessitated a different argument, supplied by Bryant and Sumners in $[Br-S]$, to establish that a 1-dimensional compact subset of a 3-dimensional hyperplane is tame in $E^4$. In this note these results are extended to

**Theorem.** A compact subset $X \subset E^4$ is tame provided, for some projection $p: E^4 \to E^3$ onto a 3-dimensional hyperplane, $\dim p(X) = 1$ and

$$\dim X \cap p^{-1}(w) \leq 0 \text{ for each } w \in E^3.$$ 

Appropriate formulations of the result for compact subsets of $E^n$ ($n \neq 4$) that have codimension $\geq 3$ are established in $[Wr, \text{Theorem 5.3}]$, the principal tool being that codimension $\geq 3$ compact subsets of $E^n$ ($n \neq 4$) are tame provided their complements are 1-ULC (see $[E]$ for a thorough discussion). The proof that follows involves applications of the previously mentioned Bryant-Sumners’ result interspersed with purely vertical moves to produce homeomorphisms that move 2-complexes off $X$.

Euclidean $n$-space is denoted by $E^n$. The product $E^4 = E^3 \times E^1$ determines a projection map $p: E^4 \to E^3$. A partition of $E^1$ is a strictly increasing function from the integers to $E^1$ that is neither bounded above nor below; it is denoted $\{a_i\}$. The partition is said to have mesh $\delta$ provided $a_{i+1} - a_i \leq \delta$ for each $i$. For a subset $W \subset E^3$, we set $W^i = W \times [a_{i-1}, a_i]$. For integers $p < q$, the collection

Received by the editors January 12, 1982.

1980 Mathematics Subject Classification. Primary 57N35, 57N45; Secondary 57N15, 57N75.

Key words and phrases. Tame embeddings, topological embeddings of compacta, 4-dimensional Euclidean space.

© 1982 American Mathematical Society

0002-9939/82/0000-0266/101.75
\[ C = \{ W^i : p \leq i \leq q \} \] is called a chain for \( W \) with respect to the partition \( \{ a_i \} \) and the length of \( C \) is set equal to \( a_q - a_{p-1} \).

**Lemma.** If \( X \subset E^4 \) is a compact subset and \( \dim(p^{-1}(s) \cap X) \leq 0 \) for some \( s \in E^3 \), then, for each \( \epsilon > 0 \), there is a neighborhood \( U \subset E^3 \) of \( s \) and a number \( \delta > 0 \) such that any chain for \( U \) with respect to a partition of mesh \( \delta \) that has length \( \geq \epsilon \) contains an element that does not meet \( X \).

**Proof.** If the lemma were false, it would be an easy matter to show that \( \dim(p^{-1}(s) \cap X) = 1 \). But this would contradict the hypothesis of the lemma, and the lemma is established.

**Proof of Theorem.** Given \( \epsilon > 0 \), use the lemma and the compactness of \( p(X) \) to produce a finite collection of closed sets \( F_1, F_2, \ldots, F_m \) that cover \( E^3 \) and a \( \delta > 0 \) such that for each \( i \) any chain for \( F_i \) with respect to a partition of mesh \( \delta \) that has length \( \geq \epsilon \) contains an element that does not meet \( X \). Assume that \( \delta < \epsilon \) and, specifying a partition by setting \( a_i = \delta \cdot i \), assume without loss of generality that \( X \subset E^3 \times [a_0, a_n] \).

Starting with a closed 2-dimensional subpolyhedron \( K \subset E^4 \), we proceed to produce a self-homeomorphism of \( E^4 \) that moves points less than a distance \( 3\epsilon \) and that moves \( K \) off \( X \). The homeomorphism arises as a composition \( H_n \circ G_n \) of self-homeomorphisms of \( E^4 \) where \( H_n \) changes only the \( E_1 \) coordinate of points and that by less than \( 2\epsilon \) and \( G_n \) moves points less than distance \( \epsilon \). Whenever \( (F_j \times [a_i, a_{i+1}] \cap X = \emptyset \), the center slice \( F_j \times \{ a_i + \delta/2 \} \) is called a safety zone. The control on the homeomorphism \( H_n \) will be achieved by not permitting \( H_n \) to move any point that lies in a safety zone, for the choice of \( F_j \)'s establishes a sufficient number of safety zones that \( H_n \) cannot move a point a distance \( \geq 2\epsilon \).

Set \( B_i = p(X \cap (E^3 \times [a_{i-1}, a_i])) \), \( A_i = \bigcup \{ F_j : F_j \cap B_i = \emptyset \} \), \( Y_i = X \cap (E^3 \times [a_0, a_i]) \), and \( X_i = p(X) \times \{ a_i \} \). The homeomorphisms \( H_n \) and \( G_n \) are the end products of an inductive construction that produces homeomorphisms \( H_i \) and \( G_i \) of \( E^4 \), \( 0 \leq i \leq n \), satisfying:

1. \( H_0 = G_0 = \text{identity homeomorphism} \);
2. \( H_i \circ G_i(K) \cap Y_i = \emptyset \) for \( 1 \leq i \leq n \);
3. \( H_i \) changes only the \( E_1 \) coordinate of points and moves no point that lies in a safety zone; and
4. \( G_i \) moves points less than \( \epsilon \).

The inductively constructed homeomorphisms \( G_i \) and \( H_i \) satisfy

\[
G_i^{-1} \circ H_i^{-1}(Y_i) \cap K = \emptyset
\]

by (2), where we set \( Y_0 = \emptyset \). The sets \( X_i \) and, therefore, \( G_i^{-1} \circ H_i^{-1}(X_i) \) are tame by \([Br-S]\). Choose a homeomorphism \( g \) of \( E^4 \) with \( g \circ G_i^{-1} \circ H_i^{-1}(X_i) \cap K = \emptyset \) so close to the identity that \( g \circ G_i^{-1} \circ H_i^{-1}(Y_i) \cap K = \emptyset \) and that, setting \( G_{i+1} = G_i \circ g^{-1} \), condition (4) is satisfied. The homeomorphism \( H_i \circ G_{i+1} \) moves \( K \) off \( Y_i \) and \( X_i \).

Choose \( 0 < d < \delta/2 \) so that

\[
H_i \circ G_{i+1}(K) \cap p(X) \times [a_i - d, a_{i+1}] = \emptyset.
\]

A homeomorphism \( h \) of \( E^4 \) equaling the identity outside \( E^3 \times [a_i - d, a_{i+1} + \delta/2] \) and on each safety zone and moving only the \( E_1 \) coordinate of points is specified as follows. Name a map \( \phi : E^3 \to [a_i, a_{i+1}] \) such that \( \phi(A_{i+1}) = a_i \) and \( \phi(B_{i+1}) = a_{i+1} \), require that \( h(x, a_i - d) = (x, a_i - d) \), \( h(x, a_i) = (x, \phi(x)) \), and...
h(x, a_i+1 + δ/2) = (x, a_i+1 + δ/2), and extend in the obvious piecewise linear manner. Finally, it is easily checked that $G_{i+1}$ and $H_{i+1} = h \circ H_i$ satisfy the appropriate conditions.

Having established that $X$ has embedding dimension at most one, we conclude that $X$ has dimension at most one; of course, applying [H-W, Theorem V17] to the restriction $p|X: X \to p(X)$ immediately reveals the latter.

REFERENCES


Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37994-1300 (Current address of J. J. Walsh)

Current address (D. G. Wright): Department of Mathematics, Utah State University, Logan, Utah 84322