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LOCALLY CONTRACTIVE AND EXPANSIVE MAPPINGS

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ABSTRACT. Locally contractive and locally expansive mappings are studied in this paper. A remetrization theorem and some fixed point theorems are proved.

1. Introduction. The classical Banach contraction principle states that if (X, d) is a complete metric space and $f: (X, d) \to (X, d)$ is a contraction (i.e. $d(f(x), f(y)) \leqslant \alpha d(x, y)$ for all $x, y \in X$, where $0 \leqslant \alpha < 1$), then f has a unique fixed point. Edelstein [4] generalized this result to mappings satisfying less restrictive assumptions such as local contractions and locally contractive mappings. Subsequently these locally contractive mappings were also studied by D. F. Bailey [1], R. D. Holmes [6], H. Rosen [8], and I. Rosenholtz [9] and many nice theorems were proved. It is our primary purpose in this paper to study these mappings and locally expansive mappings and make further contributions.

2. Basic definitions. Let (X, d) and (Y, ρ) be metric spaces. A continuous mapping $f: X \to Y$ is locally expansive (respectively, locally contractive) provided that each $x \in X$ belongs to a neighborhood N so that if y and z are distinct elements of N, $\rho(f(y), f(z)) > d(y, z)$ (respectively, $\rho(f(y), f(z)) < d(y, z)$). Furthermore, f is ε-locally expansive (respectively, contractive) if $0 < d(x, y) < \varepsilon$ implies $\rho(f(x), f(y)) > d(x, y)$ (respectively, $\rho(f(x), f(y)) < d(x, y)$). The family of all nonempty closed and bounded subsets of X is denoted by $\mathrm{CB}(X)$, and the family of all compact subsets of X is denoted by $K(X)$. If A is a subset of X and $\varepsilon > 0$, then $N(A; \varepsilon) = \{x \in X: d(x, a) < \varepsilon \text{ for some } a \in A\}$, and for $A, B \in \mathrm{CB}(X)$, the Hausdorff metric h induced by d is defined as

$$h(A, B) = \inf\{\varepsilon > 0: A \subseteq N(B; \varepsilon), B \subseteq N(A; \varepsilon)\}.$$

Equivalently, $h(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}$, where $d(x, S)$ is defined by $\inf_{y \in S} d(x, y)$. An ε-chain from x to y in X is a finite set of points $z_0, z_1, \ldots, z_n$ such that $z_0 = x$, $z_n = y$, and $d(z_i, z_{i+1}) < \varepsilon$ for all $i = 0, 1, \ldots, n-1$. A point x is said to be a fixed point of f if $x = f(x)$ (or $x \in f(x)$ if f is a set-valued mapping). Thus fixed point theorems for set-valued mappings are generalizations of their single-valued analogues.

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3. Locally contractive set-valued mappings. In this section, a remetrization theorem as well as some fixed point theorems for set-valued mappings will be proved. S. Leader [7] proved a related fixed point theorem for set-valued mappings that are local expansions. We begin with the following lemma.

LEMMA 3.1. *Let $f: (X, d) \to (\mathrm{CB}(X), h)$ be a continuous set-valued mapping. Then the mapping $\alpha: X \to R$ defined by $\alpha(x) = d(x, f(x))$ is continuous.*

PROOF. It follows immediately from the definitions that for any $x, y \in X$ and for any $A, B \in \mathrm{CB}(X)$, we have $d(x, B) \leqslant d(x, A) + h(A, B)$ and $d(x, A) \leqslant d(x, y) + d(y, A)$. Thus $d(x, f(x)) \leqslant d(x, y) + d(y, f(y)) + h(f(x), f(y))$. Hence $d(x, f(x)) - d(y, f(y)) \leqslant d(x, y) + h(f(x), f(y))$. By symmetry, we get $| d(x, f(x)) - d(y, f(y)) | \leqslant d(x, y) + h(f(x), f(y))$. It follows that $\alpha(x) = d(x, f(x))$ is continuous.

It was proved by Edelstein [4] and Rosenholtz [9] that a contractive mapping f of a compact metric space into itself has a unique fixed point. The following theorem is a generalization of their result; it is also a basic tool for later theorems.

THEOREM 3.2. *Let (X, d) be a compact metric space and $f: (X, d) \to (K(X), h)$ be contractive. Then f has a fixed point.*

PROOF. Let $\delta = \inf\{d(x, f(x)): x \in X\}$. It follows from the compactness of X and Lemma 3.1 that there exists some $x_0 \in X$ such that $\delta = d(x_0, f(x_0))$. Also, it follows from the compactness of $f(x_0)$ that there exists some $x_1 \in f(x_0)$ such that $\delta = d(x_0, x_1)$. If $\delta > 0$, then $d(x_1, f(x_1)) \leqslant h(f(x_0), f(x_1)) < d(x_0, x_1) = \delta$. That is a contradiction to the minimality of δ. Thus $d(x_0, f(x_0)) = 0$, which implies that $x_0 \in f(x_0)$ and the proof is complete.

Next, we shall prove a remetrization theorem which is a generalization of a result of Rosenholtz [9]. We shall need the following lemma, whose proof follows immediately from the definition of the Hausdorff metric.

LEMMA 3.3. *Let $A, B \in \mathrm{CB}(X)$ be such that $h(A, B) < \delta$. Then for any a in A, there exists some $b \in B$ such that $d(a, b) < \delta$.*

We are now ready for the following theorem.

THEOREM 3.4. *Let (X, d) be a compact, connected metric space and $(K(X), h)$ the space of compact subsets of X equipped with the Hausdorff metric h. Suppose $f: (X, d) \to (K(X), h)$ is locally contractive. Then there is a new metric D for X equivalent to d such that $f: (X, D) \to (K(X), H)$ is globally contractive, where H is the Hausdorff metric induced by D.*

PROOF. Since f is locally contractive, we may use compactness of X to find a positive number δ such that $h(f(x), f(y)) < d(x, y)$ whenever $0 < d(x, y) < \delta$. Now for each $p, q \in X$, define

$$D(p, q) = \inf\left\{ \sum_{j=0}^{n-1} d(x_j, x_{j+1}) \mid x_0, \ldots, x_n \text{ is a } \delta/2\text{-chain from } p \text{ to } q \right\}.$$

Observe that since X is connected, a $\delta/2$-chain always exists between each pair of points p and q. It can then be verified that D is a metric on X equivalent to d. Also, without loss of generality, we may use a bound for the number of the x_i's in calculating $D(p, q)$, say m (cf. Rosenholtz [9]). Thus

$$D(p,q) = \inf\left\{ \sum_{j=0}^{m-1} d(x_j, x_{j+1}) \mid x_0, \ldots, x_m \text{ is a } \delta/2\text{-chain from } p \text{ to } q \right\}.$$

Now by using compactness and by taking convergent subsequences at most $m-1$ times, we can find $x_0, x_1, \ldots, x_m$ such that $x_0 = p$, $x_m = q$, $d(x_j, x_{j+1}) \leqslant \delta/2$ for $j = 0, 1, \ldots, m-1$, and $D(p, q) = d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-1}, x_m)$. Since $d(x_j, x_{j+1}) \leqslant \delta/2 < \delta$, we get $h(f(x_j), f(x_{j+1})) < d(x_j, x_{j+1})$. Let $2\varepsilon_j = d(x_j, x_{j+1}) - h(f(x_j), f(x_{j+1}))$, and consequently we have $h(f(x_j), f(x_{j+1})) < d(x_j, x_{j+1}) - \varepsilon_j$ for $j = 0, 1, 2, \ldots, m-1$. Given any $y_0 \in f(x_0)$, we may use Lemma 3.3 to get $y_1 \in f(x_1)$ such that $d(y_0, y_1) < d(x_0, x_1) - \varepsilon_0$. For this $y_1 \in f(x_1)$, we may use Lemma 3.3 again to get $y_2 \in f(x_2)$ such that $d(y_1, y_2) < d(x_1, x_2) - \varepsilon_1$. Inductively, we get $y_j \in f(x_j)$ such that $d(y_{j-1}, y_j) < d(x_{j-1}, x_j) - \varepsilon_{j-1}$ for $j = 1, 2, \ldots, m$. Obviously $y_0, y_1, \ldots, y_m$ is a $\delta/2$-chain from y_0 to y_m and thus $D(y_0, y_m) \leqslant d(y_0, y_1) + d(y_1, y_2) + \cdots + d(y_{m-1}, y_m) < D(p, q) - \sum_{j=0}^{m-1} \varepsilon_j$. Let $r = D(p, q) - \sum_{j=0}^{m-1} \varepsilon_j$. We have shown that given any $y_0 \in f(x_0)$, there exists $y_m \in f(x_m)$ such that $D(y_0, y_m) < r$. Consequently $y_0 \in N_D(f(x_m); r)$, which implies that $f(x_0) \subseteq N_D(f(x_m); r)$. Similarly, we can show that $f(x_m) \subseteq N_D(f(x_0); r)$. Thus $H(f(p), f(q)) \leqslant r = (D(p, q) - \sum_{j=0}^{m-1} \varepsilon_j) < D(p, q)$. That is, $f\colon (X, D) \to (K(X), H)$ is contractive and the proof is complete.

As an application of our remetrization theorem, we obtain the following fixed point theorem which generalizes a result of Edelstein [4] and also Theorem 1 of Rosenholtz [9].

COROLLARY 3.5. *Let (X, d) be a compact connected metric space and $f\colon (X, d) \to (K(X), h)$ be a locally contractive mapping. Then f has a fixed point in X.*

PROOF. It follows from Theorem 3.4 that there exists an equivalent metric D such that $f\colon (X, D) \to (K(X), H)$ is globally contractive, and hence f has a fixed point by Theorem 3.2.

4. Locally expansive mappings. In this section, locally expansive mappings and their set-valued inverses will be investigated. One of the authors [8] recently proved a stability theorem for fixed points of a uniformly convergent sequence of open, ε-locally expansive mappings of a compact connected ANR. It is the main result of this section to show that for a Peano space, the ANR requirement can be dropped. We begin with the following lemma which is essentially due to Rosenholtz [9].

LEMMA 4.1. *Let f be an open, ε-locally expansive mapping of a compact, connected metric space (X, d) onto itself. There is a positive number $\delta < \varepsilon/2$ such that if $x, y \in X$ and $d(f(x), y) \leqslant \delta$, then there exists a unique $x_1 \in X$ such that $d(x, x_1) < \delta$ and $f(x_1) = y$.*

That open locally expansive mappings defined on compact, connected metric spaces are closely related to locally contractive set-valued mappings shall be illustrated by our next theorem.

THEOREM 4.2. *Let (X, d) be a compact connected metric space and $f\colon X \to X$ be an open, ε-locally expansive mapping. Then there is a positive number $\delta < \varepsilon/2$ such that $f^{-1}\colon (X, d) \to (K(X), h)$ is a δ-locally contractive mapping.*

PROOF. Connectedness of X implies that the open mapping f is surjective. Compactness of X and continuity of f then implies that for each x, the nonempty set $f^{-1}(x)$ is compact. Thus f^{-1} is a well-defined mapping of X into $K(X)$. We now let $F = f^{-1}$ for convenience. According to Lemma 4.1, there is a positive number $\delta < \varepsilon/2$ so that if $x, y \in X$ are such that $d(f(x), y) \leqslant \delta$, then there exists a unique $x_1 \in X$ such that $d(x, x_1) < \delta$ and $f(x_1) = y$. To show that F is δ-locally contractive, we let $x, y \in X$ be such that $0 < d(x, y) < \delta$. Next, let $r = \sup_{t \in F(x)} d(t, F(y))$. Compactness of $F(x)$ then implies that there exists some $x' \in F(x)$ such that $d(x', F(y)) = r$. Since $f(x') = x$, we have $d(f(x'), y) < \delta$. It follows now that there exists a unique $y' \in F(y)$ such that $d(x', y') < \delta$, which in turn implies that $d(x, y) = d(f(x'), f(y')) > d(x', y') \geqslant d(x', F(y)) = r$. That is, $\sup_{t \in F(x)} d(t, F(y)) < d(x, y)$. Similarly, we can prove that $\sup_{s \in F(y)} d(s, F(x)) < d(x, y)$. Hence

$$h(F(x), F(y)) = \max\left\{ \sup_{t \in F(x)} d(t, F(y)),\ \sup_{s \in F(y)} d(s, F(x)) \right\} < d(x, y).$$

Therefore $F\colon X \to K(X)$ is a δ-locally contractive mapping and the proof is complete.

The following result of Rosenholtz [**9**, **10**] now follows as an easy corollary to our theorems.

COROLLARY 4.3. *Let (X, d) be a compact connected metric space and $f\colon (X, d) \to (X, d)$ be an open locally expansive mapping. Then f has a fixed point in X.*

PROOF. Compactness of X implies that there exists an $\varepsilon > 0$ such that f is ε-locally expansive. Thus $f^{-1}\colon (X, d) \to (K(X), h)$ is δ-locally contractive for some δ by Theorem 4.2. We may now use Theorem 3.4 to obtain a metric D equivalent to d such that $f^{-1}\colon (X, D) \to (K(X), H)$ is globally contractive, and thus f^{-1} has a fixed point by Theorem 3.2. That is, there exists an x such that $x \in f^{-1}(x)$ or equivalently $f(x) = x$, and the proof is complete.

We give here additional properties of open ε-locally expansive mappings.

PROPERTY 4.4. Let (X, d) be a compact connected metric space and $f\colon X \to X$ be an open locally expansive mapping. Then f is a covering projection.

PROOF. Surjectivity of f follows from connectedness of X. We may now follow the proof of Theorem 1 in [**2**] to prove this property.

PROPERTY 4.5. Let (X, d) be a compact connected metric space and $f\colon X \to X$ be an open locally expansive mapping. Then there is an integer n such that for all $x \in X$, $\mathrm{card}[f^{-1}(x)] = n$.

PROOF. Let $A_k = \{x \in X \mid \operatorname{card}[f^{-1}(x)] = k\}$. Then it follows from a result of Eilenberg [5] that each A_k is closed as well as open. Connectedness of X now implies $X = A_n$ for some n and the property is proved.

LEMMA 4.6. *Let (X, d) be a compact connected locally connected metric space, and let f_i: $X \to X$ be open ε-locally expansive mappings for $i = 1, 2, \ldots$. Suppose the sequence $\{f_i\}_{i=1}^{\infty}$ converges uniformly to an open mapping f_0: $X \to X$. Then $f_i^{-1} \to f_0^{-1}$ uniformly as mappings of (X, d) into $(K(X), h)$.*

PROOF. Let $\eta > 0$ be given. Since f_i is ε-locally expansive for $i \geqslant 1$ and $f_i \to f_0$ uniformly, then f_0 is ε-locally noncontractive. That is, $d(x, y) < \varepsilon$ implies $d(f_0(x), f_0(y)) \geqslant d(x, y)$. According to Lemma 2 in [8], there is an open cover $\{W_\beta\}$ of X such that for each β and for $i = 0, 1, 2, \ldots,$ diam $W_\beta < \min\{\varepsilon/2, \eta\}$ and f_i maps every component of $f_i^{-1}(W_\beta)$ homeomorphically onto W_β. Let δ be a Lebesgue number for $\{W_\beta\}$. If $x, y \in X$ and $d(f_i(x), y) < \delta$, then there is some W_β containing $f_i(x)$ and y. Let C be the component of $f_i^{-1}(W_\beta)$ containing x. Then there exists a point x_1 in C with the unique property that $d(x, x_1) < \delta$ and $f_i(x_1) = y$. This shows that one delta can be found which satisfies a result like Lemma 4.1 for all f_i. Since $f_i \to f_0$ uniformly, there exists an N such that $i \geqslant N$ implies $d(f_i(x), f_0(x)) < \delta$ for all x in X. Let x_0 be an arbitrary point in X. Suppose now $i \geqslant N$ and $z \in f_i^{-1}(x_0)$. Then $d(f_i(z), f_0(z)) = d(x_0, f_0(z)) < \delta$. Then there exists a unique $x' \in X$ per z such that $d(z, x') < \delta$ and $f_0(x') = x_0$. Thus $f_i^{-1}(x_0) \subsetneq N(f_0^{-1}(x_0); \delta)$ because $f_i^{-1}(x_0)$ is finite. Similarly, $f_0^{-1}(x_0) \subsetneq N(f_i^{-1}(x_0); \delta)$ and hence $h(f_0^{-1}(x_0), f_i^{-1}(x_0)) < \delta < \eta$. This shows that f_i^{-1} converges uniformly to f_0^{-1}.

PROPERTY 4.7. With the same hypotheses as in the previous lemma, there exists an N such that $i \geqslant N$ implies $\operatorname{card}[f_i^{-1}(x)] = \operatorname{card}[f_0^{-1}(y)]$ for all x, y in X.

PROOF. Let η, δ, N, x_0, and the x''s be as in the proof of the previous lemma. Also let $m = \operatorname{card}[f_0^{-1}(x_0)]$ and $n_i = \operatorname{card}[f_i^{-1}(x_0)]$ for $i \geqslant N$. We already showed that $i \geqslant N$ implies $h(f_0^{-1}(x_0), f_i^{-1}(x_0)) < \delta$. If $m > n_i$, then it would follow from the uniqueness of the x''s that there must be some $w \in f_0^{-1}(x_0)$ for which $d(w, f_i^{-1}(x_0)) \geqslant \delta$. Therefore $h(f_0^{-1}(x_0), f_i^{-1}(x_0)) \geqslant \delta$, which is a contradiction. A similar argument shows that $m < n_i$ is also absurd. Therefore $n_i = m$, i.e. $\operatorname{card}[f_i^{-1}(x_0)] = \operatorname{card}[f_0^{-1}(x_0)]$, which in turn implies that $\operatorname{card}[f_i^{-1}(x)] = \operatorname{card}[f_0^{-1}(y)]$ for all $i \geqslant N$ and for all x, y in X by Property 4.5.

We are now ready for the main result in this section.

THEOREM 4.8. *Let (X, d) be a compact connected locally connected metric space, and let f_i: $X \to X$ be open ε-locally expansive mappings for $i = 0, 1, 2, \ldots$ such that the sequence $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f_0. Then for each fixed point a_0 of f_0, there exist fixed points a_i of f_i such that $\{a_i\}_{i=1}^{\infty}$ converges to a_0.*

PROOF. Let a_0 be any fixed point of f_0, and according to Lemma 4.1, let δ be such that $0 < \delta < \varepsilon/2$ and such that whenever $x, y \in X$ and $d(f_0(x), y) \leqslant \delta$, then there is a unique $x' \in X$ for which $d(x, x') < \delta$ and $f_0(x') = y$. Let $K = \{x \in X$: $d(x, a_0) \leqslant \delta\}$. Now for each x in K, we have $d(x, f_0(a_0)) = d(x, a_0) \leqslant \delta < \varepsilon$, and thus there exists a unique x_0 with $d(x_0, a_0) < \delta$ and $f_0(x_0) = x$. Consequently, we may define a single-valued function $g_0\colon K \to K$ by $g_0(x) = x_0$. It is easy to see that g_0 is contractive since whenever x and y are distinct points in K, we have $0 < d(x, y) \leqslant 2\delta < \varepsilon$ and therefore $d(g_0(x), g_0(y)) = d(x_0, y_0) < d(f_0(x_0), f_0(y_0)) = d(x, y)$. Now using the compactness of K and the continuity of g_0, we get $d(X - \operatorname{Int}(K), g_0(K)) = \eta > 0$. For this $\eta > 0$, it follows from Lemma 4.6 that there exists an N such that $i \geqslant N$ implies that $h(f_i^{-1}(x), f_0^{-1}(x)) < \eta$ for all x. Suppose now $i \geqslant N$ and $x \in K$ with $g_0(x) = x_0$. Then $x_0 \in f_0^{-1}(x)$, and we may use compactness of $f_i^{-1}(x)$ to get an $x_i \in f_i^{-1}(x)$ such that $d(x_0, x_i) = d(x_0, f_i^{-1}(x)) \leqslant h(f_0^{-1}(x), f_i^{-1}(x)) < \eta$. Thus $x_i \in K$ and it is unique by the ε-local expansiveness of f_i. Let $g_i\colon K \to K$ be defined by $g_i(x) = x_i$. As in the case of g_0, it can be shown that each $g_i\colon K \to K$ is contractive on the compact set K and thus has a fixed point a_i, i.e. $g_i(a_i) = a_i$ or $a_i \in f_i^{-1}(a_i)$. Thus $f_i(a_i) = a_i \in K$ for $i \geqslant N$. Since $\delta > 0$ can be chosen arbitrarily small, the theorem is proved.

If $f_i \to f_0$ pointwise on X instead of uniformly, then is the preceding theorem still true?

In Theorem 4.8, we cannot omit the assumption that f_0 is ε-locally expansive as the following example shows. Let X be the unit circle $x^2 + y^2 = 1$, and for $i \geqslant 1$, let A_i be arcs on X containing $(1, 0)$ such that $A_i \subset A_{i+1}$ and $\bigcup A_i = \operatorname{Int} A_0$, where A_0 is a semicircle with endpoints $(0, 1)$ and $(0, -1)$. For $i \geqslant 1$, there exist open ε-locally expansive mappings f_i of X such that (1) each f_i maps A_i curvilinearly onto A_0, (2) f_i has winding number 3, (3) f_i has just two fixed points $(-1, 0)$ and $(1, 0)$, and (4) f_i converges uniformly to a mapping f_0 which is the identity on A_0. Then no fixed points of the mappings f_i converge to the fixed point $(0, 1)$ of f_0.

We end with a result observed by the referee.

COROLLARY 4.9. *Let X be a compact connected locally connected metric space, $H_t\colon X \to X$, $t \in [0, 1]$ a homotopy such that for each t, H_t is an open ε-locally expansive map. Then H_0 and H_1 have exactly the same number of fixed points.*

PROOF. Define $S = \{t\colon H_0 \text{ and } H_t \text{ have a different number of fixed points}\}$. If $S \neq \varnothing$, then let $u = \inf S$. As a consequence of Theorem 4.8 and the ε-local expansiveness of H_t, it follows that if $t_n \to u$, then H_{t_n} and H_u ultimately have the same number of fixed points. Then $u \in S$ and therefore $u > 0$. But $u - u/n \to u$ implies $u - u/n \in S$ for large enough n, a contradiction.

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