INNER AMENABILITY AND FULLNESS

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Abstract. Let $G$ be a countable group which is not inner amenable. Then the II$_1$-factor $M$ is full in the following cases:

1. $M$ is given by the group measure space construction from a triple $(X, \mu, G)$ with respect to a strongly ergodic measure preserving action of $G$ on a probability space $(X, \mu)$.
2. $M$ is the crossed product of a full II$_1$-factor by $G$ with respect to an action.

1. Introduction. The set of II$_1$-factors is decomposed into two classes. The first is the set of II$_1$-factors which have property T due to Murray and von Neumann, and which contains the set of hyperfinite II$_1$-factors. The second is the set of II$_1$-factors which are called full [4]. Most known examples of II$_1$-factors are given as one of the following algebras (or composition of those): (1) the group von Neumann algebra associated with an ICC (i.e., infinite conjugacy class) group, (2) the group measure space construction algebra, or more generally, (3) the crossed product of a given von Neumann algebra by an automorphism group.

In [6], Effros introduced the notion “inner amenability” for countable groups. He showed that if a countable ICC group $G$ is not inner amenable then the group factor (associated with $G$) is full.

In this paper, we shall show that similar results hold for algebras of the above types (2) and (3), so that “noninner amenability” for groups is a desirable property in order to construct a full II$_1$-factor.

2. Full II$_1$-factor. Let $N$ be a II$_1$-factor with the canonical trace $\tau$. Then the following three statements are equivalent [4]: (4) $N$ is full, (5) $N$ has not property T, and (6) a (operator norm) bounded sequence $(x_n)$ in $N$, for which $\|x_n y - y x_n\|_2 \to 0$ for all $y \in N$, satisfies $\|x_n - \tau(x_n)1\|_2 \to 0$, where $\|x\|_2 = \tau(x^* x)^{1/2}$ for an $x \in N$.

A countable group $G$ is inner amenable if and only if there is a sequence $(\xi_n)$ in $l^2(G)$ (the Hilbert space of square summable functions on $G$) such that $\|\xi_n\|_2 = 1$, $\xi_n(1) = 0$ and $\sum_{h \in G} |\xi_n(ghg^{-1}) - \xi_n(h)|^2 \to 0$ for all $g \in G$, where $1$ is the identity of $G$ [6]. The following groups are not inner amenable: (7) the free group with two generators by [6], (8) the free product of two nontrivial groups not both of order 2, and (9) an ICC group with Kazhdan’s property T ([8]) by [1].
Let $N$ be a finite von Neumann algebra acting on a separable Hilbert space $H$, $\tau$ a faithful normal trace on $N$ such that $\tau(1) = 1$, and $G$ a countable group of $\tau$-preserving automorphisms on $N$. The action of $G$ is said to be ergodic on $N$ if $N^G = \{x \in N; g(x) = x \text{ for all } g \in G\} = C_1$, and strongly ergodic on $N$ if a bounded sequence $(x_n)$ in $N$ for which $\|g(x_n) - x_n\|_2 \to 0$ for all $g \in G$, satisfies necessarily $\|x_n - \tau(x_n)1\|_2 \to 0$ (cf. [5]). A strongly ergodic action is ergodic. If $G$ has property T, then an ergodic action of $G$ is strongly ergodic [2]. Put

$$(\pi(a)\xi)(g) = g^{-1}(a)\xi(g) \quad \text{and} \quad (\nu(g)\xi)(h) = \xi(g^{-1}h)$$

$$(a \in N, g, h \in G, \xi \in l^2(G, H)),$$

where $l^2(G, H)$ is the Hilbert space of square summable $H$-valued functions on $G$. Then $\pi$ (resp. $\nu$) is a representation of $N$ (resp. $G$) on $l^2(G, H)$ such that $\nu(g)\pi(a)\nu(g)^* = \pi(g(a))$ for all $g \in G$ and $a \in N$. The crossed product $M$ of $N$ by $G$ is the von Neumann algebra generated by $\pi(N)$ and $\nu(G)$. Let $e$ be the faithful normal expectation of $M$ onto $\pi(N)$ such that $e(\nu(g)) = 0$ for $g \neq 1$ (see [7], for example). Then $M$ is a finite von Neumann algebra with a faithful normal trace $\tau \cdot e$ and each $x \in M$ has a unique expansion $x = \sum_{g \in G} x(g)\nu(g)$ ($x(g) \in \pi(N)$) for all $g \in G$) in the sense of $\|\cdot\|_2$-metric convergence.

It is known that ergodicity is essential to the group measure space construction for a factor. The strong ergodicity is necessary for the group measure space construction of a full $\text{II}_1$-factor [3].

**Theorem.** Let $N$ be a finite von Neumann algebra with a faithful normal trace $\tau$ such that $\tau(1) = 1$, $G$ a countable group of $\tau$-preserving automorphisms of $N$ and $M$ be the crossed product of $N$ by $G$. Assume that $G$ is not inner amenable. Then

(i) A sequence $(x_n)$ in $M$, for which $\|x_n\|_2 = 1$ for all $n$ and $\|x_n \nu(g) - \nu(g)x_n\|_2 \to 0$ for all $g \in G$, satisfies $\|x_n - e(x_n)\|_2 \to 0$. In particular, $\nu(G) \cap M = \pi(N^G)$.

(ii) If the action of $G$ is strongly ergodic on $N$, then $M$ is a full $\text{II}_1$-factor.

**Proof.** (i) Let $(x_n)$ be a sequence in $M$ for which $\|x_n\|_2 = 1$ for all $n$ and $\|x_n \nu(g) - \nu(g)x_n\|_2 \to 0$ for all $g \in G$. Let $x_n = \sum_{g \in G} x(g)\nu(g)$ ($x_n(g) \in \pi(N)$) be the Fourier expansion of $x_n$. For each $n$, put $\xi_n = \sum_{g \in G} \|x_n(g)\|_2 \delta(g)$, where $\delta(g)$ is the characteristic function of $\{g\}$. Then $(\xi_n)$ is a sequence of unit vectors in $l^2(G)$ and satisfies

$$\sum_{h \in G} |\xi_n(ghg^{-1}) - \xi_n(h)|^2 = \sum_{h \in G} \|x_n(ghg^{-1})\|_2 - \|x_n(h)\|_2^2$$

$$= \sum_{h \in G} \|x_n(ghg^{-1})\|_2 - \|\nu(g)x_n(h)\nu(g)^*\|_2^2$$

$$\leq \sum_{h \in G} \|x_n(ghg^{-1}) - \nu(g)x_n(h)\nu(g)^*\|_2^2$$

$$= \|x_n\nu(g) - \nu(g)x_n\|_2^2 \to 0, \text{ for all } g \in G.$$

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If \( \| \xi_n - \xi_n(1)\delta(1) \|_2 \) does not converge to zero, there are an \( \alpha > 0 \) and a subsequence \( (\xi_{n(k)}) \) of \( (\xi_n) \) such that \( \| \xi_{n(k)} - \xi_{n(k)}(1)\delta(1) \|_2 \geq \alpha \) for all \( k \). For each \( k \), put
\[
\xi_k = (\xi_{n(k)} - \xi_{n(k)}(1)\delta(1)) / \| \xi_{n(k)} - \xi_{n(k)}(1)\delta(1) \|_2.
\]
Then the sequence \( (\xi_k) \) satisfies that \( \| \xi_k \|_2 = 1 \), \( \xi_k(1) = 0 \) and
\[
\sum_{h \in G} |\xi_k(ghg^{-1}) - \xi_k(h)|^2 \to 0.
\]
This contradicts the noninner amenability of \( G \). Hence \( \| \xi_n - \xi_n(1)\delta(1) \|_2 \to 0 \).

Therefore,
\[
\|x_n - e(x_n)\|_2^2 = \sum_{g \neq 1} \|x_n(g)\|_2^2 = \|\xi_n - \xi_n(1)\delta(1)\|_2^2 \to 0.
\]

(ii) If the action of \( G \) is ergodic on \( N \), then by (i) \( M' \cap M \subset \pi(N) \cap M = \pi(N^G) \) = Cl. Hence \( M \) is a finite factor. If \( G \) is not ICC, there is an \( h (\neq 1) \in G \) for which \( \{ghg^{-1}; g \in G\} \) is a finite set. Put \( z = \sum_{g \in G} v(ghg^{-1}) \). Then \( e(z) = 0 \) and \( 0 \neq z \in \pi(G) \cap M = Cl. \) This is a contradiction. Hence \( G \) is an ICC group.

Therefore \( M \) contains a II\(_1\)-factor \( v(G)' \), so that \( M \) is a II\(_1\)-factor. Assume that \( M \) is not full. Since \( M \) has property \( \Gamma \), there is a sequence \( (x_n) \) of unitaries in \( M \) for which \( \|zx_n - x_nz\|_2 \to 0 \) for all \( z \in M \) and \( \tau(e(x_n)) = 0 \) for all \( n \). Since
\[
\|x_n v(g) - v(g)x_n\|_2 \to 0 \quad \text{for all } g \in G,
\]
the sequence \( (x_n) \) satisfies, by (i), \( \|x_n - e(x_n)\|_2 \to 0 \). The expectation \( e \) satisfies \( e(v(g)v(g)^*) = v(g)e(\gamma)v(g)^* \) for all \( g \in G \) and \( \gamma \in M \). Therefore,
\[
\|v(g)e(x_n) - e(x_n)v(g)\|_2 \to 0 \quad \text{for all } g \in G.
\]
Since the action of \( G \) is strongly ergodic on \( N \), we have that
\[
\|e(x_n)\|_2 = \|e(x_n) - \tau(e(x_n))\|_2 \to 0.
\]
This contradicts the assumption that all \( x_n \) are unitaries. Thus \( M \) is a full II\(_1\)-factor.

In the case that \( N \) is a full II\(_1\)-factor, the assumption of strong ergodicity for the action of \( G \) is not necessary.

**Corollary.** Let \( N \) be a full II\(_1\)-factor and \( G \) a countable group of automorphisms of \( N \). If \( G \) is not inner amenable, then the crossed product \( M \) of \( N \) by \( G \) is a full II\(_1\)-factor.

**Proof.** Let \( \tau \) be the canonical trace on \( N \). Since \( N \) is a factor, by (i) in the Theorem, \( M' \cap M \subset \pi(N)' \cap M = \pi(N^G) \cap M = Cl. \) Hence \( M \) is a finite factor. Since \( N \) is type II, \( M \) is a II\(_1\)-factor. If \( M \) is not full, there is a sequence \( (x_n) \) of unitaries in \( M \) for which \( \tau(e(x_n)) = 0 \) for all \( n \) and \( \|x_nz - x_nz\|_2 \to 0 \) for all \( z \in M \). By the same proof as (ii) in the Theorem, \( \|x_n - e(x_n)\|_2 \to 0 \). On the other hand, the bounded sequence \( e(x_n) \) satisfies \( \|ye(x_n) - e(x_n)y\|_2 \to 0 \) for all \( y \in \pi(N) \), because \( e \) is an expectation of \( M \) onto \( \pi(N) \). Therefore,
\[
\|e(x_n)\|_2 = \|e(x_n) - \tau(e(x_n))\|_2 \to 0,
\]
because \( N \) is full. This is a contradiction. Thus \( M \) is a full II\(_1\)-factor.
References


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