SPACES FOR WHICH THE GENERALIZED CANTOR SPACE $2^J$ IS A REMAINDER

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Abstract. Let $X$ be a locally compact noncompact space, $m$ be an infinite cardinal and $|J| = m$. Let $F(X)$ be the algebra of continuous functions from $X$ into $\mathbb{R}$ which have finite range outside of an open set with compact closure and let $I(X) = \{g \in F(X) : g \text{ vanishes outside of an open set with compact closure}\}$. Conditions on $R(X) = F(X)/I(X)$ and internal conditions are obtained which characterize when $X$ has $2^J$ as a remainder.

1. Introduction. Throughout this paper all spaces are assumed to be completely regular and Hausdorff. We let LC denote the class of all locally compact and noncompact spaces. A compactification of a space $X$ is a compact space which contains $X$ as a dense subspace and a remainder of $X$ is any $aX \setminus X$ where $aX$ is a compactification of $X$. If $aX$ and $bX$ are two compactifications of $X$, then $aX \leq bX$ if there is a continuous function $g : aX \to bX$ such that $g(x) = x$ for each $x \in X$. For a set $A$ let $|A|$ denote the cardinality of $A$.

Recently Hatzenbuhler and Mattson [HM] have obtained an internal characterization which characterizes when a given space $X \in \text{LC}$ has every compact metric space as a remainder. The condition given by them assures that if $X$ satisfies this condition the Cantor space $2^\mathbb{N}$ is a remainder of $X$, where $\mathbb{N}$ is the set of natural numbers and $2$ is the discrete space $\{0, 1\}$. Their result then follows from the fact that every compact metric space is a continuous image of $2^\mathbb{N}$. It is thus natural to ask when for a given cardinal $m$ and a space $X \in \text{LC}$, $2^J$ is a remainder of $X$ where $|J| = m$. In this connection we briefly recall the construction of the Freudenthal compactification.

Definition 1.1. Let $X$ be a space. An ordered pair $(G, H)$ is called an $f$-pair in $X$ if $G$ and $H$ are disjoint open subsets of $X$ and $X \setminus (G \cup H)$ is compact.

Let $X \in \text{LC}$. For subsets $A$ and $B$ of $X$, we define the relation $\delta$ by $A \delta B$ if and only if there is an $f$-pair $(G, H)$ in $X$ such that $\text{cl}_X A \subseteq G$ and $\text{cl}_X B \subseteq H$. It is well known that $\delta$ is a compatible proximity relation on $X$ and the Samuel compactification $fX$ corresponding to this proximity relation is the Freudenthal compactification of $X$ [W, 41.2, 41B]. It is known [R] that $fX \setminus X$ is zero dimensional and if $aX$ is any compactification of $X$ such that $aX \setminus X$ is zero dimensional, then $aX \leq fX$. By a zero dimensional space, we mean a space which has a basis consisting of clopen, i.e., both
closed and open sets. It follows in particular that if \( 2^J \) is a remainder of \( X \), then \( 2^J \) is a continuous image of \( fX \setminus X \).

In \([E]\) Efimov introduces the concept of a dyadic family of power \( m \) and proves that \([0, 1]^J \), where \( |J| = m \), is a continuous image of a compact space \( Y \) if and only if \( Y \) has a dyadic family of power \( m \). In the view of these observations, one naturally expects to obtain an internal characterization, via the Freudenthal compactification which characterizes when a space \( X \in LC \) has \( 2^J \) as a remainder. For this purpose we slightly modify Efimov's definition of a dyadic family.

**Definition 1.2.** Let \( X \) be a space, \( m \) be a cardinal and \( J \) be a set with \( |J| = m \). A family \( \{(U^{-1}_j, U^+_j) : j \in J\} \) consisting of \( f \)-pairs in \( X \) is called a *dyadic family of power \( m \)* in \( X \) if for every finite collection of distinct elements \( j_1, \ldots, j_n \) of \( J \) and any finite sequence \( i_1, \ldots, i_n \) in \( \{-1, 1\} \), \( \text{cl}_X(U^{-1}_{j_1} \cap \cdots \cap U^{-1}_{j_n}) \) is not compact.

We prove that if \( X \in LC \), \( m \) is an infinite cardinal and \( |J| = m \), then \( 2^J \) is a remainder of \( X \) if and only if \( X \) has a dyadic family of power \( m \). We also give an algebraic characterization which is equivalent to the one given above.

For a space \( X \) let \( \kappa(X) \) denote the set of all open subsets of \( X \) with compact closure in \( X \). Let \( C(X) \) be the algebra of all continuous functions from \( X \) into the set of real numbers \( \mathbb{R} \).

**Definition 1.3.** Let \( X \in LC \). We set

\[
F(X) = \{ g \in C(X) : g(X \setminus V) \text{ is finite for some } V \in \kappa(X) \},
\]

\[
I(X) = \{ g \in C(X) : X \setminus g^{-1}(0) \in \kappa(X) \}.
\]

Clearly \( F(X) \) is a subalgebra of \( C(X) \) and \( I(X) \) is an ideal in \( F(X) \). We set \( R(X) = F(X)/I(X) \).

We prove that if \( X \in LC \), then the structure space \( \text{Max} F(X) \) of \( F(X) \) can be identified with \( fX \). We also prove that if \( m \) is an infinite cardinal and \( |J| = m \), then \( 2^J \) is a remainder of \( X \) if and only if the group of units of \( R(X) \) has a subgroup \( G \) of cardinality \( m \) such that \( g^2 = 1 \) for every \( g \in G \) and \( G \) is linearly independent over \( \mathbb{R} \).

2. **Structure space of \( F(X) \).** If \( g \) is a function from a set \( A \) into \( \mathbb{R} \), \( Z(g) = \{ a \in A : g(a) = 0 \} \) is the zero set of \( g \). If \( \alpha \in \mathbb{R} \), then we will use the same notation \( \alpha \) to denote the constant function from \( A \) into \( \mathbb{R} \) whose value is \( \alpha \). Let \( X \in LC \). It is easy to verify that \( \{ X \setminus Z(g) : g \in I(X) \} = B_X \) forms a base for open sets in \( X \). For \( x \in X \) let us define \( M_x = \{ g \in F(X) : g(x) = 0 \} \). Then \( M_x \) is a maximal ideal in \( F(X) \) and if \( x \) and \( y \) are distinct elements of \( X \), then \( M_x \neq M_y \) since \( B_X \) forms a base for open sets in \( X \). A maximal ideal \( M \) of \( F(X) \) is, by definition, *fixed* if \( M = M_x \) for some \( x \in X \), otherwise \( M \) is *free*.

Let \( S \) be any commutative ring with identity. The structure space of \( S \) is the set of all maximal ideals \( \text{Max} S \) of \( S \) topologized by taking the sets of the form \( E(s) = \{ M \in \text{Max} S : s \in M \} \) as a base for closed sets \([GJ, 7M]\). \( \text{Max} S \) with this topology is compact but not necessarily Hausdorff. If \( S \) is von Neumann regular, then \( \text{Max} S \) is Hausdorff. Recall that \( S \) is von Neumann regular if for each \( a \in S \), there exists \( b \in S \) such that \( a^2b = a \).
Proposition 2.1. Let $X \in \text{LC}$ and $M \in \text{Max } F(X)$.
(a) $R(X)$ is von Neumann regular and hence $\text{Max } R(X)$ is a compact Hausdorff space.
(b) $M$ is free if and only if $I(X) \subseteq M$.

Proof. (a) Let $g \in F(X)$ and $V \in \kappa(X)$ be such that $g(X \setminus V) = \{\alpha_1, \ldots, \alpha_n\}$. Note that $X \setminus V \neq \emptyset$ since $X$ is not compact. Let $K = \text{Z}(g)$ and $L = g^{-1}(\{\alpha_i : \alpha_i \neq 0\})$. Then $K$ and $L$ are disjoint zero sets in $X$. Thus there is a continuous function $h: X \to [0, 1]$ such that $K \subseteq \text{int}_X Z(h) \subseteq Z(h) = A$ and $L \subseteq h^{-1}(1) \subseteq X \setminus \text{int}_X Z(h) = B$. $A$ and $B$ are closed sets in $X$ and $A \cup B = X$. Define $w: X \to \mathbb{R}$ by $w(x) = 0$ if $x \in A$ and $w(x) = h(x)/g(x)$ if $x \in B$. $w$ is well defined and continuous. Note that $w(X \setminus V) \subseteq \{0\} \cup \{1/\alpha_i : \alpha_i \neq 0\}$. It follows that $w \in F(X)$ and $g^2w - g \in I(X)$. Thus $R(X)$ is von Neumann regular.

(b) Let $x \in X$ and $V \in \kappa(X)$ be a neighbourhood of $x$. Then there is a continuous function $g: X \to [0, 1]$ such that $g(x) = 1$ and $g(X \setminus V) = \{0\}$. Thus $g \in I(X) \setminus M_x$. Consequently a fixed maximal ideal cannot contain $I(X)$. Now, let $M$ be a free maximal ideal. Suppose that there exists $g \in I(X) \setminus M$. Since $M$ is maximal, then $gk - 1 \in M$ for some $k \in F(X)$. Let $V = X \setminus \text{Z}(g) \in \kappa(X)$. For each $x \in \text{cl}_X V$, $M \setminus M_x \neq \emptyset$. Thus $\text{cl}_X V \subseteq \bigcup \{X \setminus \text{Z}(t) : t \in M\}$. Since $\text{cl}_X V$ is compact, then there are $t_1, \ldots, t_n \in M$ such that $\text{cl}_X V \subseteq \bigcup \{X \setminus \text{Z}(t_i) : i = 1, \ldots, n\}$. Let $t = t_1^2 + \cdots + t_n^2 \in M$. Then $\text{cl}_X V \subseteq X \setminus \text{Z}(t)$. There is an $0 < \varepsilon < 1$ such that $t(x) \geq \varepsilon$ for each $x \in \text{cl}_X V$. If $r = (gk - 1)^2 + t$, then $r \in M$ and $r(x) \geq \varepsilon$ for all $x \in X$. Thus $M$ contains an invertible element, a contradiction. So $M \supseteq I(X)$.

We have already seen that the function $x \to M_x$ sets up a one-to-one correspondence between $X$ and the fixed maximal ideals in $F(X)$. Hence $X$ already constitutes an index set for the fixed maximal ideals in $F(X)$. We enlarge it to an index set $fX$ for $\text{Max } F(X)$, so that $\text{Max } F(X) = \{M_y : y \in fX\}$ and for distinct $y, z \in fX$, $M_y \neq M_z$. For $g \in F(X)$ let $F(g) = \{y \in fX : g \in M_y\}$. If $\theta: fX \to \text{Max } F(X)$ is the function defined by $\theta(y) = M_y$, then $\theta^{-1}(F(g)) = F(g)$ for $g \in F(X)$. Thus $\{F(g) : g \in F(X)\}$ forms a base for closed sets of a topology on $fX$ and with this topology $fX$ is compact and homeomorphic to $\text{Max } F(X)$. Theorem 2.2. Let $X \in \text{LC}$. For a subset $A$ of $X$ let $A^* = \text{cl}_{fX} A \setminus X$.

(a) $fX$ is a compactification of $X$ and $fX \setminus X$ is homeomorphic to $\text{Max } R(X)$.
(b) Each function $g \in F(X)$ has a unique continuous extension $g^*: fX \to R$ and $F(g) = Z(g^*)$.
(c) Let $(G, H)$ be an $f$-pair in $X$. Then there exist $g \in F(X)$ and $W \in \kappa(X)$ such that $X \setminus (G \cup H) \subseteq W$, $\text{cl}_X G \setminus W \subseteq g^{-1}(-1)$ and $\text{cl}_X H \setminus W \subseteq g^{-1}(1)$. If $U$ is any open subset of $X$, then $(G \cap U)' = G' \cap U'$. In particular $G'$ and $H'$ are disjoint clopen subsets of $X'$ whose union is $X'$.
(d) $fX$ is the Freudenthal compactification of $X$.

Proof. (a) We have already observed that $fX$ is compact. By 2.1(b) if $g \in I(X)$, then $fX \setminus F(g) = X \setminus Z(g)$. It follows that the topology on $X$ coincides with the
subspace topology inherited from $fX$. Also, $X = \bigcup \{X \setminus Z(g): g \in I(X)\} = \bigcup \{fX \setminus F(g): g \in I(X)\}$. So $X$ is an open subspace of $fX$. If $h \in F(X)$ and $fX \setminus F(h) \neq \emptyset$, then $h \neq 0$. So $x \in fX \setminus F(h)$ for some $x \in X$. Thus $X$ is dense in $fX$.

We now proceed to show that $fX$ is Hausdorff. Let $y, z \in fX$ and $y \neq z$. If $y, z \in X$, then $y$ and $z$ can be separated by open sets in $X$ and hence in $fX$ since $X$ is open in $fX$. Thus suppose without loss of generality that $z \notin X$. Let $g \in M_y \setminus M_z$. By 2.1(a), $g^2h - g \in I(X) \subseteq M_z$ for some $h \in F(X)$. Let $b = gh - 1$. Since $M_z$ is prime, then $b \in M_z$. Note also that $b \notin M_y$. Let $V = X \setminus Z(gb) \in k(X)$. Let $W \in k(X)$ be such that $cl_X V \subseteq W$. There is a continuous function $u: X \rightarrow [0,1]$ such that $cl_X V \subseteq Z(u)$ and $X \setminus W \subseteq Z(u - 1)$. Note that $u - 1 \in I(X)$ and $ugb = 0$. So $z \in fX \setminus F(ug)$, $y \in fX \setminus F(b)$ and $F(ug) \cup F(b) = fX$. This proves that $fX$ is Hausdorff. To see that $X'$ and $P = \text{Max } R(X)$ are homeomorphic, consider the natural homomorphism $\phi: F(X) \rightarrow R(X)$. $\phi$ induces a bijection $\phi': P \rightarrow X'$ defined by $\phi'(N) = z$ if and only if $\phi^{-1}(N) = M_z$. $\phi'$ is continuous since $(\phi)^{-1}(F(g) \setminus X) = E(g + I(X))$ for every $g \in F(X)$. Both $X'$ and $P$ are compact Hausdorff, thus $\phi'$ is a homeomorphism.

(b) Let $g \in F(X)$ and $V \in k(X)$ be such that $g(X \setminus V) = \{a_1, \ldots, a_n\}$ where $a_i \neq a_j$ for $i \neq j$. Let $h = (g - a_1) \cdots (g - a_n)$. Since $X \setminus Z(h) \subseteq V$, then $h \in I(X)$. So $fX = cl_X V \cup F(h) = cl_X V \cup F(g - a_1) \cup \cdots \cup F(g - a_n)$. If $i \neq j$, then $0 \neq a_i - a_j \notin M_y$. For any $y \in fX$. Thus $F(g - a_i) \cap F(g - a_j) = \emptyset$ for $i \neq j$. Moreover if $x \in cl_X V \cap F(g - a_i)$, then $g(x) = a_i$. We define $g^e$: $fX \rightarrow R$ by $g^e(y) = a_i$ if $y \in F(g - a_i)$ and $g^e(y) = g(y)$ if $y \in cl_X V$. Then $g^e$ is well defined and continuous. It is routine to verify that $F(g) = Z(g^e)$ and $g^e$ extends $g$.

(c) Let $L = X \setminus (G \cup H)$. Let $V, W \in k(X)$ be such that $L \subseteq W \subseteq cl_X W \subseteq V$. Then $C = cl_X G \cap cl_X V \setminus W$ and $D = cl_X H \cap cl_X V \setminus W$ are disjoint closed subsets of the compact space $cl_X V$. Thus there is a continuous function $h: cl_X V \rightarrow [-1,1]$ such that $C \subseteq h^{-1}(1)$ and $D \subseteq h^{-1}(-1)$. Let $g: X \rightarrow [-1,1]$ be defined by $g(x) = -1$ if $x \in cl_X G \setminus W$, $g(x) = 1$ if $x \in cl_X H \setminus W$ and $g(x) = h(x)$ if $x \in cl_X V$. It is easy to see that $g$ satisfies the required properties. Since $W \in k(X)$ and $L$ is compact, then $g^e(G) \subseteq \{-1\}$, $g^e(H) \subseteq \{1\}$ and $G' \cup H' = X'$. So $G'$ and $H'$ are disjoint clopen subsets of $X'$ whose union is $X'$. Now let $U$ be any open subset of $X$. Suppose that $y \in U' \cap G' \setminus (U \cap G)'$ for some $y \in X'$. There is an open neighbourhood $S$ of $y$ in $fX$ such that $S \cap (L \cup H' \cup (U \cap G)) = \emptyset$. Then $S \cap U \subseteq H$ and consequently $y \in (S \cap U)' \subseteq H'$, a contradiction. So $(U \cap G)' = U' \cap G'$.

(d) We must show that if $A, B \subseteq X$, then $cl_{fX} A \cap cl_{fX} B = \emptyset$ if and only if there is an $f$-pair $(G, H)$ in $X$ such that $cl_X A \subseteq G$ and $cl_X B \subseteq H$. “if” part is clear from (c). Thus suppose that $A, B \subseteq X$ and $cl_{fX} A \cap cl_{fX} B = \emptyset$. $\{F(g): g \in F(X)\} = \{Z(g^e): g \in F(X)\}$ is a basis for closed sets in $fX$ and it is closed under finite intersections. Hence there exist $g, h \in F(X)$ such that $cl_{fX} A \subseteq F(g), cl_{fX} B \subseteq F(h)$ and $F(g) \cap F(h) = Z(g^e) \cap Z(h^e) = Z((g^2 + h^2)^e) = \emptyset$. This implies that $g^2 + h^2$ is a unit in $F(X)$. Let $w = g^2/(g^2 + h^2)$. Then $0 < w(x) < 1$ for all $x \in X$. Let $V \in k(X)$ be such that $w(X \setminus V) = \{a_1, \ldots, a_n\}$. Pick a real number $0 < \alpha < 1$ such that $\alpha \neq \alpha_i$ for $i = 1, \ldots, n$. Let $G = \{x \in X: w(x) < \alpha\}$ and $H = \{x \in X: w(x) > \alpha\}$. Then $cl_X A \subseteq G$, $cl_X B \subseteq H$, $G \cap H = \emptyset$ and $X \setminus (G \cup H) \subseteq (w^e)^{-1}(\alpha) \subseteq X$. Thus $(G, H)$ is an $f$-pair with the required properties.
3. $2^J$ as a remainder. Let $D$ be the discrete space $\{-1, 1\}$. Then $2^J$ is homeomorphic to $D^J$. In what follows, it will be more convenient to work with $D$ than $2$ and we will do so. We first state a lemma which follows easily from Theorem 2.2(c) by induction.

**Lemma 3.1.** Let $X \in LC$ and $(G_1, H_1), \ldots, (G_n, H_n)$ be a finite sequence of $f$-pairs in $X$. Then $(G_1 \cap \cdots \cap G_n)' = G_1' \cap \cdots \cap G_n'$, where for a subset $A$ of $X$, $A' = \text{cl}_X A \setminus X$.

We now state our main result.

**Theorem 3.2.** Let $X \in LC$, $m$ be an infinite cardinal and $J$ be a set of cardinality $m$. Then the following are equivalent.

(a) $X$ has a dyadic family of power $m$.

(b) The group of units of $R(X)$ has a subgroup $G$ of cardinality $m$ such that $g^2 = 1$ for all $g \in G$ and $G$ is linearly independent over $R$.

(c) $D^J$ is a remainder of $X$.

**Proof.** (a) implies (b). Let $\Delta = \{(U_j^{-1}, U_j^1) : j \in J\}$ be a dyadic family of power $m$ in $X$. For each $j \in J$ we pick a function $g_j \in F(X)$ and a member $V_j$ of $\kappa(X)$ such that for $i \in D \cap (x U_j \setminus V_j) \subseteq g_j^{-1}(i)$. The existence of $g_j$ is guaranteed by 2.2(c). Let $r_j = g_j + I(X), j \in J$. Since

$$X \setminus Z_1 = L_j \cup \text{cl}_X V_j$$

where $L_j = X \setminus (U_j^{-1} \cup U_j^1)$, then $r_j^2 = 1$ for all $j \in J$. Let $G$ be the group generated by $\{r_j : j \in J\}$. Then clearly $r^2 = 1$ for all $r \in G$. Let $j_1, \ldots, j_n$ be distinct elements of $J$ and $H$ be the subgroup of $G$ generated by $A = \{r_j : k = 1, \ldots, n\}$. Let $T$ be the linear subspace of $R(X)$ spanned by $H$. Since $|A| \leq n$ and $r_j^2 = 1$ for all $j \in J$, then $|H| \leq 2^n$. It follows that $\dim_R T \leq 2^n$. For an $n$-tuple $\eta = (\eta_1, \ldots, \eta_n) \in D^n$ let $e_\eta \in F(X)$ be defined by

$$e_\eta = 2^{-n}(1 + \eta_1 g_{j_1}) \cdots (1 + \eta_n g_{j_n}).$$

Let $P_k = \text{cl}_X V_{j_k} \cup L_{j_k}, 1 \leq k \leq n$, and $P = P_1 \cup \cdots \cup P_n$. We claim that $e_\eta \notin I(X)$. For suppose that $e_\eta \in I(X)$ and $V = X \setminus Z(e_\eta)$. Then $P \cup \text{cl}_X V$ is compact. Thus if $Q = U_j^{\eta_j} \cap \cdots \cap U_{j_n}^{\eta_n}$, then $\hat{Q} = Q \cup P \cup \text{cl}_X V \neq \emptyset$ since $\Delta$ is a dyadic family. Let $x \in \hat{Q}$. Then for $1 \leq k \leq n$, $x \in U_{j_k}^{\eta_k} \setminus V_{j_k}$ which implies that $2^{-1}(1 + \eta_k g_{j_k}(x)) = 2^{-1}(1 + \eta_k^2) = 1$. Thus $0 = e(x) = 1$, a contradiction. Let $\hat{\eta} = e_\eta + I(X)$. Then $\hat{\eta}$ is a nonzero element of $T$. If $\eta$ and $\rho$ are distinct $n$-tuples in $D^n$, then it is easy to see that $\hat{\eta}^* \hat{\rho} = 0$ and $\hat{\eta}$ is an idempotent in $T$. Thus $\{\hat{\eta} : \eta \in D^n\}$ is a linearly independent subset of $T$ containing exactly $2^n$ elements. This shows that $\dim_R T = 2^n$. Thus $|H| = 2^n$ and $H$ is linearly independent over $R$. Since every finite subset $B$ of $G$ is also a subset of a subgroup $H$ described as above, then $B$ is linearly independent. Also the argument given above shows that $|\{r_j : j \in J\}| = m$. Thus $|G| = m$.

(b) implies (c). $G$ is a 2-group and so it has a basis. This means that there is a subset $B$ of $G$ such that if $b_1, \ldots, b_n$ are distinct elements of $B$, then $b_1 \cdots b_n \neq 1$ and each element of $G$ can be written as a finite product of elements in $B$. Since $|G| = m$ and $m$ is an infinite cardinal, then $|B| = m$. We define a function $\psi : \text{Max } R(X) \to D^B$ as follows: Let $P = \text{Max } R(X)$ and $M \in P$. If $b \in B$ then $b^2 - 1 \in M$. Thus either
b - 1 \in M \text{ or } b + 1 \in M. \text{ But } 2 \not\in M \text{ and so only one of } b - 1 \text{ or } b + 1 \text{ may be in } M. \text{ We define } \psi(M)(b) = -1 \text{ if } b + 1 \in M \text{ and } \psi(M)(b) = 1 \text{ if } b - 1 \in M. \text{ Let } \pi_b : D^B \rightarrow D \text{ denote the } b^{th} \text{ projection. If } b_1, \ldots, b_n \text{ are distinct elements of } B \text{ and } i_1, \ldots, i_n \in D, \text{ then } \psi^{-1}(\pi_{b_1}^{-1}(i_1) \cap \cdots \cap \pi_{b_n}^{-1}(i_n)) = P \setminus E((b_1 + i_1) \cdots (b_n + i_n)). \text{ Hence } \psi \text{ is continuous. Moreover } \psi \text{ is onto, for let } x \in D^B. \text{ The ideal } T \text{ of } R(X) \text{ generated by } \{b - x(b): b \in B\} \text{ is distinct from } R(X). \text{ For otherwise there are elements } r_1, \ldots, r_n \in R(X) \text{ and } b_1, \ldots, b_n \in B \text{ such that } b_i's \text{ are distinct and}
\begin{equation}
r_1(b_1 - x(b_1)) + \cdots + r_n(b_n - x(b_n)) = 1.
\end{equation}
\text{Let } r \text{ be the product of the elements } b_k + x(b_k), 1 \leq k \leq n. \text{ Then multiplying both sides of (i) by } r \text{ we obtain } r = 0. \text{ Since } B \text{ is a basis for } G, \text{ then } r \text{ is a linear combination of the pairwise distinct elements } 1, b_1, \ldots, b_n, b_1b_2, \ldots, b_1b_2 \cdots b_n \text{ of } G \text{ with } 1 \text{ having the coefficient } \mp 1. \text{ This is a contradiction as } G \text{ is linearly independent over } R. \text{ So } T \neq R(X). \text{ If } M \text{ is a maximal ideal containing } T, \text{ then } \psi(M) = x. \text{ So } \psi \text{ is onto. It follows that } D^B \text{ is a continuous image of } P \text{ and so of } f_X \times X \text{ by } 2.2(a). \text{ Now, utilizing upper semicontinuous decompositions as in } [M] \text{ we can construct a compactification } aX \text{ of } X \text{ with } aX \times X = D^B.

(c) implies (a). \text{ Let } aX \text{ be a compactification of } X \text{ such that } aX \times X = D^J. \text{ Then } fX \supseteq aX \text{ since } D^J \text{ is zero dimensional. Let } \phi : fX \times X \rightarrow D^J \text{ be a continuous surjection. For } j \in J \text{ and } i \in D, \text{ let us set } W_j^i = \phi^{-1}(\pi_j^{-1}(i)). \text{ Then } W_j^{-1} \text{ and } W_j^1 \text{ are disjoint clopen sets in } X' \text{ whose union is } X'. \text{ Let } V_j^{-1} \text{ and } V_j^1 \text{ be disjoint open neighbourhoods of } W_j^{-1} \text{ and } W_j^1, \text{ respectively, in } fX. \text{ Let } U_j^i = X \cap V_j^i \text{ for } i \in D \text{ and } j \in J. \text{ If } T = V_j^{-1} \cup V_j^1, \text{ then } (X \setminus T \cap X) \cap T = \emptyset \text{ and } T \text{ is an open neighbourhood of } X'. \text{ Thus } cl_{fX}(X \times X \cap T) \subseteq X, \text{ i.e., } X \times X \cap T \text{ is compact. So } (U_j^{-1}, U_j^1) \text{ is an } f\text{-pair in } X \text{ for each } j \in J. \text{ Let } \Delta \text{ be the set of all these pairs. If } j_1, \ldots, j_k \text{ are distinct elements of } J \text{ and } i_1, \ldots, i_n \in D, \text{ then by Lemma 3.1, } (\cap U_j^{i_k})' = \cap (U_j^{i_k})' = \cap W_j^{i_k} = \phi^{-1}(\cap \pi_j^{-1}(i_k)) \neq \emptyset \text{ where the intersections are taken over } k, 1 \leq k \leq n. \text{ Thus } \Delta \text{ is a dyadic family of power } m \text{ in } X.

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