A LOWER BOUND ON THE CESÀRO OPERATOR

RUSSELL LYONS

If the sequence \( a = (a_n)_{n=0}^{\infty} \in l^2 \), i.e., \( \|a\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \), define \( Sa \) as the sequence of averages

\[
\left( \frac{1}{n+1} \sum_{k=0}^{n} a_k \right)_{n=0}^{\infty}.
\]

It follows easily from the Marcinkiewicz Interpolation Theorem that \( S \) is a bounded operator from \( l^2 \) to \( l^2 \); this can also be proved directly using the Cauchy-Buniakowski-Schwarz inequality [1]. \( S \) is known as the Cesàro operator. \( S \) is, of course, not bounded below, but the following property does hold, confirming a conjecture of Allen Shields and Sheldon Axler.

**Theorem.** If \( a_0 \geq a_1 \geq \cdots \geq 0 \), \( a = (a_n)_{n=0}^{\infty} \), then \( \|Sa\|^2 \geq \pi^2 \|a\|^2 / 6 \), with equality if and only if \( a_1 = a_2 = \cdots = 0 \).

**Proof.** If we expand the squares and group terms, we find that

\[
\|Sa\|^2 = \frac{\pi^2}{6} \|a\|^2 - \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \frac{1}{k^2} \right) a_n^2 + \sum_{n=1}^{\infty} \left( 2 \sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) \sum_{j=0}^{n-1} a_j a_n
\]

\[
\geq \frac{\pi^2}{6} \|a\|^2 + \sum_{n=1}^{\infty} \left( 2n \sum_{k=n+1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \right) a_n^2.
\]

Since

\[
2n \sum_{k=n+1}^{\infty} \frac{1}{k^2} > 2n \int_{n+1}^{\infty} \frac{dx}{x^2} = \frac{2n}{n+1} \geq \sum_{k=1}^{n} \frac{1}{k^2},
\]

the result follows.

**References**


Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109