RELATIVE NORMAL COMPLEMENTS IN FINITE GROUPS

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Abstract. $(G, H, H_0, \pi)$ denotes the following configuration: $H$ and $H_0$ are the subgroups of the finite group $G$ with $H_0 \leq H$ and $\pi$ is the set of primes dividing $(H : H_0)$. For $(G, H, H_0, \pi)$ we consider conditions (A), (B0), and (C): (A) Any two $\pi$-elements of $H - H_0$ which are $G$-conjugate are $H$-conjugate. (B0) For each $\pi$-element $x \in H - H_0$, $C_G(x) = I(x)C_H(x)$ where $I(x)$ is a normal $\pi'$-subgroup of $C_G(x)$. (C) $|\langle H - H_0 \rangle^{G, \pi}| = (G : H) |H - H_0|$. We show that if $(G, H, H_0, \pi)$ satisfies (B0) and (C), or (A) and (B0), and if $H/H_0$ is solvable, then there is a unique relative normal complement $G_0$ of $H$ over $H_0$.

All groups in this paper are finite. Given a group $G$ with subgroups $H_0, H,$ and $G_0$ such that $H_0 \leq H$, we call $G_0$ a relative normal complement of $H$ over $H_0$ if $G_0 \leq G, G = G_0H$ and $H_0 = G_0 \cap H$.

Let $\pi(G)$ denote the set of primes dividing $|G|$. If $\pi$ is a set of primes, then the complementary set of primes will be denoted by $\pi'$. A group $G$ is a $\pi$-group if $\pi(G) \subseteq \pi$. If $x \in G$, then $x$ is a $\pi$-element if $\langle x \rangle$ is a $\pi$-group. Every element $x$ of $G$ has a unique decomposition $x = x_\pi x_{\pi'} = x_\pi x_{\pi'}$ into a $\pi$-element $x_\pi$ and a $\pi'$-element $x_{\pi'}$. Further, $x_\pi$ and $x_{\pi'}$ are powers of $x$. If $x$ and $y$ are elements of a subgroup $K$ of $G$, then $x$ and $y$ belong to the same $\pi$-section of $K$ if their $\pi$-parts $x_\pi$ and $y_\pi$ are $K$-conjugate. If $S$ is a subset of $G$, then $S^{G, \pi}$ denotes the union of all $\pi$-sections of $G$ which intersect $S$.

We let $(G, H, H_0, \pi)$ denote the following configuration: $G$ is a finite group with subgroups $H$ and $H_0$ such that $H_0 \leq H$ and $\pi = \pi(H/H_0)$. For $(G, H, H_0, \pi)$ we consider the following conditions:

(A) Any two $\pi$-elements of $H - H_0$ which are $G$-conjugate are $H$-conjugate.

(B0) For each $\pi$-element $x \in H - H_0$, $C_G(x) = I(x)C_H(x)$ where $I(x)$ is a normal $\pi'$-subgroup of $C_G(x)$.

(C) $|\langle H - H_0 \rangle^{G, \pi}| = (G : H) |H - H_0|$.

Leonard [2] has shown that if $(G, H, H_0, \pi)$ satisfies conditions (B0) and (C) and $\pi = \{p\}$ or $I(x)$ is always a Hall $\pi'$-subgroup of $C_G(x)$, then there is a unique relative normal complement $G_0$ of $H$ over $H_0$ and $G_0 = G - (H - H_0)^{G, \pi}$. If $(G, H, H_0, \pi)$ satisfies conditions (A) and (B0) and $\pi = \{p\}$, then Reynolds [3] has shown that there is a unique relative normal complement $G_0$ of $H$ over $H_0$ and $G_0 = G - (H - H_0)^{G, \pi}$. In this paper, we prove two generalizations of these theorems.
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Theorem 1. If \((G, H, H_0, \pi)\) satisfies conditions \((B_0)\) and \((C)\) and \(H/H_0\) is solvable, then there is a unique relative normal complement \(G_0\) of \(H\) over \(H_0\) and \(G_0 = G - (H - H_0)^G_\pi\).

Theorem 2. If \((G, H, H_0, \pi)\) satisfies conditions \((A)\) and \((B_0)\) and \(H/H_0\) is solvable, then there is a unique relative normal complement \(G_0\) of \(H\) over \(H_0\) and \(G_0 = G - (H - H_0)^G_\pi\).

We omit stating explicitly the obvious corollaries which follow from Theorems 1 and 2 by replacing “\(H/H_0\) is solvable” by “\(\pi\) is a set of odd primes”.

If a set \(R\) is a disjoint union of \(S\) and \(T\), we write \(R = S \cup T\).

Lemma 1. Assume \((G, H, H_0, \pi)\) satisfies condition \((B_0)\), \(K\) is a subgroup of \(H\) containing \(\pi^1\) and \(\pi^2 \subseteq \pi\). Then

\[
(K - H_0)^{G_\pi} \cup (H - K)^{G_\pi} \subseteq (H - H_0)^{G_\pi}.
\]

Proof. Assume \(w \in (H - K)^{G_\pi} \cup (H - H_0)^{G_\pi}\), then \(w = w_n w_\pi\) and \(w_\pi = w_\pi w_n\), where \(\pi_1 = \pi - \pi_2\). Further, \(w_n = x^\pi\) where \(x\) is a \(\pi_2\)-element in \(H - K\) or \(x\) is a \(\pi_2\)-element in \(K - H_0\). Since \(K \supseteq H_0\), \(x\) is a \(\pi\)-element in \(H - H_0\); hence \(w_\pi w_\pi^{-1} \in C_G(x) = I(x)C_H(x)\). Let \(\tilde{w} = (w_\pi)^{-1}\), then \(\tilde{w}I(x) = hi(x)\) where \(h \in C_H(x)\). Since \(I(x)\) is a normal \(\pi\)-subgroup of \(C_G(x)\), \(h\) is a \(\pi\)-element, \(h\) may be taken to be a \(\pi\)-element of \(H\). Now \(\langle \tilde{w} \rangle I(x) = \langle h \rangle I(x)\) and Theorem 6.2.1 of \([1]\) imply that \(\langle \tilde{w} \rangle = \langle h \rangle^i\) where \(i \in I(x)\). It follows that \(h^{\pi^i} = \tilde{w}\) for some integer \(i\), and \(h^{\pi^i} = \tilde{w}^{\pi^i} = x\). Since \(i \in C_G(x)\), \(h\) is a \(\pi\)-element of \(H - H_0\) and \(\tilde{w}\) is conjugate to an element of \(H - H_0\). Thus, \(w \in (H - H_0)^{G_\pi}\).

Proof of Theorem 1. If \((G, H, H_0, \pi)\) satisfies conditions \((B_0)\) and \((C)\), then Leonard showed in §3 of \([2]\), that \(H - H_0\) is a union of \(\pi\)-sections of \(H\) and condition \((A)\) is satisfied.

Let \(G\) be a minimal counterexample to Theorem 1. Theorem 1.2 of \([2]\) implies that \(\pi\) contains more than one prime. Since \(H/H_0\) is solvable, there is a prime \(p \in \pi\) such that \(H\) contains an abelian \(p\)-factor group. Thus, \(H\) contains a normal subgroup \(H_1\) of index \(p\) and \(H_1 \supseteq H_0\). Let \(D = H - H_1\), \(x \in D\) and \(y \in H\) where \(x_p\) and \(y_p\) are \(H\)-conjugate. Then \(x_{p'}\) and \(y_{p'}\) \(H_1\)-conjugate. Thus, \(y_p \in H - H_1\) and \(y \in H - H_1\). Therefore, \(D\) is a union of \(p\)-sections of \(H\). If \(x\) and \(y\) are \(p\)-elements in \(D\) which are \(G\)-conjugate, then \(x\) and \(y\) are \(G\)-conjugate \(\pi\)-elements in \(H - H_0\). Condition \((A)\) implies \(x\) and \(y\) are \(H\)-conjugate. If \(x\) is a \(p\)-element in \(D\), then \(C_G(x) = O_p(C_G(x))C_H(x)\) follows from condition \((B_0)\). Applying Theorem 2 of \([3]\) to \(G, H,\) and \(H_1\), we see that \(G\) has a unique relative normal complement \(G_1\) of \(H\) over \(H_1\) and \(G_1 = G - (H - H_1)^G_\pi\). Now \(G = G_1 \cup (H - H_1)^G_\pi\) and \((G_1 : H_1) = (G : H)\) yield

\[
| (H - H_1)^{G_\pi} | = | G_1 | = | G_1 | (\{(H : H_1) - 1\}) = (G_1 : H_1) | H - H_1 | .
\]

Let \(R = G_1 \cap (H - H_0)^{G_\pi}\) and \(\pi_1 = \pi(H_1/H_0)\). We will show that \((G_1, H_1, H_0, \pi_1)\) satisfies the hypothesis of Theorem 1 and \(R = (H_1 - H_0)^{G_1_\pi_1}\). It follows from Lemma 1, \(\pi_1 \subseteq \pi\) and \(H_1 \supseteq H_0\), that \((H_1 - H_0)^{G_1_\pi_1} \subseteq R\). Let \(w \in R\), then \(w = x^\pi\) where \(x\) is a \(\pi\)-element in \(H - H_0\). We may write \(g = hy\) where \(h \in H\).
and $y \in G_1$. Let $\tilde{w} = w_{\pi_1}^{-1}$ and $\tilde{x} = x^h$, then $\tilde{w} \in G_1$; hence, $\tilde{x} \in G_1 \cap (H - H_0) = H_1 - H_0$. Since $y \in G_1$, it follows that $w \in (H_1 - H_0)^{G_\pi}$. If $\pi = \pi_1$, then $\tilde{x} = \tilde{x}_{\pi_1} \tilde{x}_p$. Now $H_1/H_0$ a $p$-group implies that $\tilde{x}_p \in H_0$. Thus, $\tilde{x}_{\pi_1} \in H_1 - H_0$ and $\tilde{w}_{\pi_1} = \tilde{x}_{\pi_1}$. Again $y \in G_1$ so $w \in (H_1 - H_0)^{G_\pi}$. Therefore, $R = (H_1 - H_0)^{G_\pi}$. Lemma 1, applied to $H_1$ and $p$, implies that $(H_1 - H_0)^{G_\pi} \subseteq (H - H_0)^{G_\pi}$. Using $G = G_1 \cup (H - H_1)^{G_\pi}$, we see that $(H - H_0)^{G_\pi} = R \cup (H - H_1)^{G_\pi}$. Now condition (C) and $(G : H) = (G_1 : H_1)$ yield

$$|(H_1 - H_0)^{G_\pi}| = |R| = \left| (H - H_0)^{G_\pi} \right| - \left| (H - H_1)^{G_\pi} \right| = (G_1 : H_1)(|H - H_0| - |H - H_1|) = (G_1 : H_1)|H_1 - H_0|.$$

Hence, $(G_1, H_1, H_0, \pi_1)$ satisfies condition (C).

Let $x$ be a $\pi_1$-element in $H_1 - H_0$. Then $x$ is a $\pi$-element in $H - H_0$; hence, $C_{G_\pi}(x) = I(x)C_{H_\pi}(x)$. Since $\pi_1 \subseteq \pi$ and $(G : G_1) = p$, $I(x) \subseteq G_1$ and $I(x)$ is a normal $\pi_1$-subgroup of $C_{G_\pi}(x)$. Further, $C_{G_\pi}(x) = I(x)C_{H_\pi}(x) \cap G_1 = I(x)(C_{H_\pi}(x) \cap G_1) = I(x)C_{H_\pi}(x)$ so that condition (B) is satisfied by $(G_1, H_1, H_0, \pi_1)$.

Clearly $H_1/H_0$ is solvable since $H/H_0$ is. The minimality of $|G|$ now implies that $G_1$ has a unique relative normal complement $G_0$ of $H_1$ over $H_0$, namely $G_0 = G_1 - (H_1 - H_0)^{G_\pi}$. It has been shown that $(H_1 - H_0)^{G_\pi} = R$ is a normal subset of $G$. Thus, $G_0$ is a normal subgroup of $G$. Clearly, $G_0$ is a relative normal complement of $H$ over $H_0$. Proposition 2.2 of [2] now implies that $G_0$ is unique and $G_0 = G - (H - H_0)^{G_\pi}$.

**Proof of Theorem 2.** Assume $(G, H, H_0, \pi)$ satisfies the hypothesis of Theorem 2 and let $D = H - H_0$. Assume $x \in D$ and $y \in H$ where $x_\pi$ is $H$-conjugate to $y_\pi$. Then $x_\pi$ and $y_\pi$ are $H$-conjugate to $y_\pi$. Thus, $x_\pi$ and $y_\pi$ are $H$-conjugate to $y_\pi$. Hence, $y \in H - H_0$; hence, $x \in H - H_0$ and $D$ is a union of $\pi$-sections of $H$.

If $S$ is a $\pi$-section of $G$ and $S \cap D \neq \emptyset$, let $x$ and $y \in S \cap D$. Then $x$ and $y \in H - H_0$, and in particular, $x_\pi$ and $y_\pi$ are $H$-conjugate to $y_\pi$. Using condition (A), we see that $x_\pi$ and $y_\pi$ are $H$-conjugate; hence, $x$ and $y$ lie in the same $\pi$-section of $H$. Therefore, $S \cap D = \emptyset$ or $S \cap D$ is a $\pi$-section of $H$. Lemma 3.4 of [2] now implies that

$$|(H - H_0)^{G_\pi}| = |D^{G_\pi}| = (G : H) |D| = (G : H)|H - H_0|.$$

Thus, $(G, H, H_0, \pi)$ satisfies conditions (B) and (C), and $H/H_0$ is solvable. Theorem 2 follows from Theorem 1.

**References**


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