AUTOMATIC CONTINUITY OF MEASURABLE GROUP HOMOMORPHISMS

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Abstract. It is well known that a measurable homomorphism from a locally compact group $G$ to a topological group $Y$ must be continuous if $Y$ is either separable or $\sigma$-compact. In this work it is shown that the above requirement on $Y$ can be somewhat relaxed and it is shown inter alia that a measurable homomorphism from a locally compact group to a locally compact abelian group will always be continuous. In addition, it is shown that if $H$ is a nonopen subgroup of a locally compact group, then under a variety of circumstances, some union of cosets of $H$ must fail to be measurable.

If $f$ is a function from a locally compact group $G$ to a topological space $Y$, and if $m_G$ is the left Haar measure of $G$, then we say that $f$ is measurable if $f^{-1}(U)$ is an $m_G$-measurable subset of $G$ whenever $U$ is open in $Y$. Certainly, every continuous function must be measurable. When $Y$ is a topological group and $f$ is a homomorphism, then there is a variety of circumstances under which the measurability of $f$ is sufficient to guarantee its continuity. For example, in [2, Theorem 22.18], it is shown that a measurable homomorphism $f$ will certainly be continuous if the group $Y$ is either separable or $\sigma$-compact. It is our purpose here to show that this condition on $Y$ can sometimes be replaced by the weaker condition that $Y$ have an open subgroup which is either separable or $\sigma$-compact. The latter condition is especially interesting because every locally compact group has an open $\sigma$-compact subgroup.

As examples of results that can be obtained, we state the following:

**Theorem 1.** Let $f$ be a measurable homomorphism from a locally compact group $G$ to a locally compact group $Y$ and suppose that at least one of the groups $G$ and $Y$ is abelian. Then $f$ is continuous.²

**Theorem 2.** Let $f$ be a measurable homomorphism from a locally compact group $G$ to a topological group $Y$. Suppose $Y$ has an open normal subgroup $Z$ which is either separable or $\sigma$-compact and suppose that at least one of the groups $G$ and $Y/Z$ is solvable. Then $f$ is continuous.

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²The referee has kindly pointed out that if both $G$ and $Y$ are assumed abelian, Theorem 1 is proved as Theorem 3.1 in [1].
Theorem 2 can be stated more sharply as follows:

**Theorem 3.** Let \( f \) be a measurable homomorphism from a locally compact group \( G \) to a topological group \( Y \) which has an open normal subgroup \( Z \) which is either separable or \( \sigma \)-compact. Let \( H = f^{-1}(Z) \) and suppose that the group \( H/H \) is solvable. Then \( f \) is continuous (and therefore, \( H = H') \).

**Theorem 4.** A measurable homomorphism from a locally compact group to a discrete group is continuous if and only if its kernel is closed.

The following lemma suggests the method of proof of the above theorems.

**Lemma 1.** Let \( f \) be a measurable homomorphism from a locally compact group \( G \) to a topological group \( Y \) and suppose that \( Y \) has an open subgroup \( Z \) which is either separable or \( \sigma \)-compact. Then a (necessary and) sufficient condition for \( f \) to be continuous is that the group \( f^{-1}(Z) \) be open in \( G \).

**Proof.** The necessity is trivial. Now let \( H = f^{-1}(Z) \) and assume that \( H \) is an open subgroup of \( G \). Since the left Haar measure of \( H \) is the restriction to \( H \) of the left Haar measure of \( G \), we see that for subsets of \( H \), measurability with respect to the Haar measures of \( G \) and \( H \) is the same. Therefore, the restriction of \( f \) to \( H \) is measurable from \( H \) to \( Z \) and is therefore continuous by [2, Theorem 22.18]. So \( f \) being continuous at the identity \( 0 \) of \( G \) must be continuous.

In view of Lemma 1, the above theorems will follow as soon as we have determined sufficient conditions for the group \( H = f^{-1}(Z) \) to be open in \( G \). In this case the union of any family of cosets of \( H \) must be measurable, being the image under \( f^{-1} \) of the union of a family of cosets of \( Z \) in \( Y \). This prompts us to make the following definition.

**Definition.** A subgroup \( H \) of a locally compact group \( G \) is said to be totally measurable if given any family of cosets of \( H \), the union of the family is measurable with respect to the left Haar measure of \( G \).

We conjecture that only an open subgroup can be totally measurable. The following theorem is a step in this direction and in view of Lemma 1, all the above theorems follow at once from it.

**Theorem 5.** Let \( H \) be a totally measurable normal subgroup of a locally compact group \( G \). Then \( H \) is open in \( G \) and the group \( H/H \) is equal to its commutator subgroup. A fortiori, if \( H/H \) is solvable (for example, if \( G \) is solvable), then \( H = H' \) and consequently, \( H \) is open in \( G \).

In order to prove Theorem 5, we shall need a few technical results. In what follows, if \( H \) is a normal subgroup of a locally compact group \( G \), \( \phi \) will denote the natural homomorphism from \( G \) to \( G/H \). Where appropriate, \( m_G \), \( m_H \) and \( m_{G/H} \) will denote left Haar measures of \( G \), \( H \) and \( G/H \) respectively. All groups will be written additively.

**Lemma 2.** Let \( m \) be a regular measure on a locally compact Hausdorff space \( X \) and let \( F \) be a family of nonnegative lower semicontinuous functions on \( X \) such that every two members of \( F \) have a common upper bound in \( F \). For each \( x \) in \( X \), let \( g(x) = \sup \{ f(x) \mid f \in F \} \). Then we have \( \int_X g \, dm = \sup \{ \int_X f \, dm \mid f \in F \} \).
PROOF. One can prove this directly but this result is essentially [2, Theorem 11.33].

LEMMA 3. Let $H$ be a closed normal subgroup of a locally compact group $G$. Let $A$ be a subset of $G$ and suppose that $m_G(A) = 0$. Define $B = \{ x \in G \mid m_H((-x + A) \cap H) > 0 \}$. Then $B$ is a union of cosets of $H$ and this family $\phi(B)$ of cosets has $m_{G/H}$ measure zero, i.e. $m_{G/H}(\phi(B)) = 0$.

PROOF. It is clear that $B$ is a union of cosets of $H$. Now following [2, §§15.21–15.23] (and see also [3, §2.7.3]), we adjust the Haar measures such that for every $f$ continuous on $G$ with compact support we have

\[
\int_G f(x) \, dm_G(x) = \int_{G/H} \int_H f(x + y) \, dm_H(y) \, dm_{G/H}(\phi(x)).
\]

Using Lemma 2, one can show easily that (*) holds whenever $f$ is nonnegative and lower semicontinuous. In particular, (*) holds for $f = \chi_U$ whenever $U$ is open in $G$ and has finite measure. For each natural $n$, choose an open neighborhood $U_n$ of $A$ such that $m_G(U_n) < 1/n$. If $E$ is the intersection of the sets $U_n$, then the dominated convergence theorem implies that

\[
0 = m_G(E) = \int_{G/H} \int_H \chi_E(x + y) \, dm_H(y) \, dm_{G/H}(\phi(x))
\]

and from this equality, the lemma follows easily.

LEMMA 4. Let $H$ be a closed normal subgroup of a locally compact group $G$. Let $A$ be the union of a family of cosets of $H$ and suppose that $m_G(A) = 0$. Then $m_{G/H}(\phi(A)) = 0$.

PROOF. This follows at once from Lemma 3 since for every $x$ in $A$, $H \subseteq -x + A$ and so $m_H((-x + A) \cap H) \geq m_H(H) > 0$.

LEMMA 5. Let $H$ be a closed normal subgroup of a locally compact $\sigma$-compact group $G$. Let $A$ be an $m_G$-measurable subset of $G$ and suppose that $A$ is the union of a family of cosets of $H$. Then $\phi(A)$ is $m_{G/H}$-measurable.

PROOF. Since $m_G$ is $\sigma$-finite on $A$, we can write $A$ in the form $\bigcup_{n=0}^\infty A_n$ where $m_G(A_0) = 0$ and $A_n$ is compact for every $n \geq 1$. Define $B_n = A_n + H$ for $n \geq 1$ and $B_0 = A \setminus \bigcup_{n=1}^\infty B_n$. Then for $n \geq 1$, since $\phi(B_n) = \phi(A_n)$, we see that $\phi(B_n)$ is a compact subset of $G/H$. Furthermore, since $B_0 \subseteq A_0$, we have $m_G(B_0) = 0$. But $B_0$ is a union of cosets of $H$ and we conclude from Lemma 4 that $m_{G/H}(\phi(B_0)) = 0$. Therefore $\phi(A)$ being the union of the sets $\phi(B_n)$, must be $m_{G/H}$-measurable.

We are now ready to prove the special case of Theorem 5 that occurs when $H$ is closed.

LEMMA 6. Let $H$ be a totally measurable closed normal subgroup of a locally compact group $G$. Then $H$ is open.

\[\text{It is not necessary to assume that } G \text{ is } \sigma\text{-compact. However, the proof in the general case is a little more technical and the lemma as stated here is sufficient for our purposes.}\]
Proof. We assume that \( H \) is a closed normal subgroup of \( G \) but that \( H \) is not open. To prove the lemma, we shall show that some union of cosets of \( H \) must fail to be \( m_G \)-measurable. Choose an open \( \sigma \)-compact subgroup \( G_1 \) of \( G \) and define \( H_1 = H \cap G_1 \). Since the interior (in \( G \)) of \( H \) is empty, we see that \( H_1 \) is not an open subgroup of the locally compact group \( G_1 \). Therefore the group \( G_1/H_1 \) is not discrete and it follows from [2, §16.13] that \( G_1/H_1 \) has a subset \( E \) which is not \( m_{G_1/H_1} \)-measurable. \( E \) is a family of cosets of \( H_1 \) in \( G_1 \) and the union \( A \) of these cosets is not \( m_{G_1} \)-measurable by Lemma 5. Since \( G_1 \) is open in \( G \), it follows that \( A \) is not \( m_G \)-measurable. But every coset of \( H_1 \) in \( G_1 \) is the intersection with \( G_1 \) of a coset of \( H \) in \( G \). Therefore, \( A \) is of the form \( A = B \cap G_1 \) where \( B \) is a union of cosets of \( H \) in \( G \); and clearly, the set \( B \) cannot be \( m_G \)-measurable.

To complete the proof of Theorem 5, we must now concern ourselves with nonclosed subgroups.

**Lemma 5.** Let \( H \) be a totally measurable dense normal subgroup of a locally compact group \( G \). Then the group \( G/H \) is equal to its commutator subgroup. A fortiori, if \( G/H \) is solvable, we have \( H = G \).

Proof. We may write the commutator subgroup of \( G/H \) in the form \( K/H \) where \( K \) is a subgroup of \( G \) and to obtain a contradiction we assume that \( K \) is a proper subgroup. The group \( G/K \) is abelian. If \( K \) has countable index in \( G \), define \( L = K \). Otherwise, as in [2, §16.13], choose a subgroup \( L/K \) of \( G/K \) which has countably infinite index in \( G/K \). In either event, \( L \) is a proper subgroup of \( G \) and has countable index in \( G \) and \( L \) includes \( H \). Being of countable index in \( G \), \( L \) cannot be \( m_G \)-locally null. Also, \( L \) is the union of a family of cosets of \( H \) and therefore, \( L \) is \( m_G \)-measurable and it follows that \( L \) is open and therefore closed. Therefore, since \( H \) is dense in \( G \), we have \( L = G \) in contradiction to our choice of \( L \) as a proper subgroup.

Proof of Theorem 5. Since \( H \) is normal in \( G \), so is \( \overline{H} \). Furthermore, since every coset of \( \overline{H} \) is the union of a family of cosets of \( H \), we see that \( \overline{H} \) is totally measurable and we conclude from Lemma 6 that \( \overline{H} \) is an open subgroup of \( G \). Therefore, for subsets of \( \overline{H} \), \( m_G \)-measurability and \( m_{\overline{H}} \)-measurability are the same and we conclude that \( H \) is totally measurable as a subgroup of the locally compact group \( \overline{H} \). The result now follows from an application of Lemma 7.

Some results about nonnormal subgroups. The question as to whether a nonnormal totally measurable subgroup must be open, seems very hard to answer. If \( H \) is a nonnormal subgroup of \( G \), we can still speak of the set \( G/H \) of left cosets of \( H \) and the natural map \( \phi \) from \( G \) to \( G/H \). With the quotient topology, \( G/H \) is locally compact, and is Hausdorff if \( H \) is closed, and discrete if \( H \) is open. Since an analogue of Lemma 7 for nonnormal subgroups seems hopeless, we shall assume that \( H \) is closed. Although it is meaningless to speak of a Haar measure on \( G/H \), there may still be a measure \( m_{G/H} \) on \( G/H \) that has much of the behaviour of a left Haar measure and for which the equality (*) in Lemma 3 holds. By [2, Theorem 15.24], the condition for the existence of this measure \( m_{G/H} \) is that the modular functions of \( G \) and \( H \) should agree at every point of \( H \). This would hold, for
example, if $H$ were normal in $G$, and also if $G$ and $H$ were unimodular. We recall that every compact group is unimodular.

With this measure on $G/H$, we can repeat the proofs of Lemmas 3, 4 and 5; the chief difference being that we must now refer to left cosets of $H$. Unfortunately, Lemma 6 is more troublesome as it is not obvious that there is a subset $E$ of $G/H$ which is not $m_{G/H}$-measurable. The following result, which depends on the continuum hypothesis, is the best we can do.

**Theorem 6.** Let $H$ be a totally measurable closed subgroup of a $\sigma$-compact locally compact group $G$ of cardinality $|G| = \aleph_1 = 2^{\aleph_0}$ and suppose that the modular functions of $G$ and $H$ agree at every point of $H$. Then $H$ is an open subgroup of $G$.

**Proof.** It is a well-known consequence of the continuum hypothesis that there is no nontrivial finite continuous measure on the $\sigma$-algebra of all subsets of a set of cardinality $\aleph_1$. Furthermore, $m_{G/H}$ being nontrivial, must be positive (and finite) on some compact set. To obtain a contradiction, assume that $H$ is not open in $G$. Then it is easy to see that $m_{G/H}$ is continuous and consequently, $G/H$ must have a subset $E$ which is not $m_{G/H}$-measurable. Therefore, by an analogue of Lemma 5, $E + H$ is a non-$m_{G}$-measurable union of left cosets of $H$ and we have a contradiction.

For compact groups, more can be proved. Once again, we require the continuum hypothesis.

**Theorem 7.** Let $H$ be a totally measurable subgroup of a compact group $G$ of cardinality $|G| = \aleph_1 = 2^{\aleph_0}$. Then $H$ is open.

**Proof.** To obtain a contradiction, we assume that $H$ is not open. Then $m_{G}(H) = 0$. Given any subset $E$ of $G/H$, define $\mu(E) = m_{G}(\phi^{-1}(E))$. Clearly, $\mu$ is a finite, continuous, nontrivial measure on the $\sigma$-algebra of all subsets of the set $G/H$ and we have reached a contradiction.

Finally, we conclude

**Theorem 8.** Let $G$ be a compact group of cardinality $|G| = \aleph_1 = 2^{\aleph_0}$ and let $f$ be a measurable homomorphism from $G$ to a topological group $Y$ which has an open subgroup which is either separable or $\sigma$-compact. Then $f$ is continuous.

**References**


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