A NOTE ON NEIGHBOURHOODS OF UNIVALENT FUNCTIONS

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ABSTRACT. Using a notion of neighbourhood of analytic functions due to Stephan Ruscheweyh we examine conditions under which neighbourhoods of a certain class of convex functions are included in a class of starlike functions.

Introduction. Let $A$ denote the class of analytic functions $f$ in the unit disk $E$: \{z | |z| < 1\} with $f(0) = 0$, $f'(0) = 1$. For $f(z) := z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $\delta > 0$ Ruscheweyh has defined the neighbourhood $N_\delta(f)$ as follows:

$$N_\delta(f) := \left\{ g(z) := z + \sum_{k=2}^{\infty} b_k z^k \left| \sum_{k=2}^{\infty} k |a_k - b_k| \leq \delta \right. \right\}.$$ 

He has shown in [1] among other results that if $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in C$ the following result is true:

$$N_{\delta_n}(f) \subset S^*, \quad \delta_n = 2^{-2/n},$$ 

where $C(S^*)$ denotes the class of normalized convex (starlike) univalent functions in $A$. He also asked if a similar result would hold if we replace $S^*$ by the class

$$T := \left\{ g \in S^* \left| z \frac{g'(z)}{g(z)} - 1 < 1, z \in E \right. \right\}$$ 

and $C$ by the class

$$\tilde{T} := \left\{ g \in C \left| z \frac{g''(z)}{g'(z)} < 1, z \in E \right. \right\}.$$

We prove

**Theorem 1.** Let $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in T$. Then $N_{\delta_n}(f) \subset \tilde{T}$, $\delta_n = e^{-1/n}$.

Let $S^*_\alpha$ (0 \leq \alpha < 1) denotes the class \{ $g \in S^* | \text{Re}[z(g'(z)/g(z)) > \alpha, z \in E]$ \}. An analogue of this class with respect to $T$ is the class

$$T_r := \left\{ g \in T \left| z \frac{g'(z)}{g(z)} - 1 < r, z \in E \right. \right\}, \quad 0 < r \leq 1.$$ 

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In [1] Ruscheweyh has shown that for no $\alpha \in [0, 1)$ is there a positive $\delta$ such that $N_\delta(S^*) \subset S^*$. For the class $T$, the situation is quite different as shown by

**Theorem 2.** Let $g(z) := z + \sum_{k=n+1}^\infty a_k z^k \in T$, $0 \leq r < 1$. Then $N_\delta(g) \subset T$, $\delta_n = e^{-r/n}(1 - r)$.

The boundaries of $\{w \in \hat{C} | \text{Re}[w] > 0\}$ and of $\{w \in \hat{C} | \text{Re}[w] > \alpha\}$ are not disjoint whereas those of $\{w \in \hat{C} | |w - 1| < 1\}$ and of $\{w \in \hat{C} | |w - 1| < r\}$ are; this is one of the reasons for the difference between the two situations. Nevertheless Theorem 2 is still interesting since the value for $\delta_n$ is best possible.

Concerning this question of boundaries we can prove

**Theorem 3.** Let $f \in T$ and $D := \{zf'(z)/f(z) | z \in E\}$ be such that there is $w \in \overline{D}$ with $|w - 1| = 1$. Then for no $\delta > 0$ we have $N_\delta(f) \subset T$.

It should be noted that no similar result holds if the class $T$ is replaced by the class $S^*$: in fact for $f(z) = z/(-1 - z) \in C \subset S^*$ we have $N_{1/4}(f) \subset S^*$ even though the region $D = \{w \in C | \text{Re}[w] > \frac{1}{2}\}$ is such that the point at infinity belongs to both $\overline{D}$ and $\{w \in \hat{C} | \text{Re}[w] > \alpha\}$.

**Proof of Theorem 1.** It was established in [2] that for $f(z) := z + \sum_{k=2}^\infty a_k z^k \in T$ we have the estimate $|z| e^{-|z|/n} \leq |f(z)| \leq |z| e^{|z|/n}$, using the same method it is very easy to show that for $f(z) := z + \sum_{k=n+1}^\infty a_k z^k \in T$ the estimate

$$|z| e^{-|z|/n} \leq |f(z)| \leq |z| e^{|z|/n}$$

is true and sharp as seen from the function $f(z) := ze^{z/n}$. We also remark that $f(z) \in \hat{T} \Rightarrow zf'(z) \in T$ so that we obtain for $f(z) := z + \sum_{k=n+1}^\infty a_k z^k \in \hat{T}$ for the following estimate

$$e^{-|z|/n} \leq |f'(z)| \leq e^{|z|/n}$$

and the sharpness is established by looking at the function $f(z) := \int_0^z e^{u/n} du$.

We also remark the following: a function $g(z) \in A$ belongs to the class $T$ iff for every $\theta \in [0, 2\pi)$ we have

$$z \frac{g'(z)}{g(z)} - 1 \neq e^{i\theta}, \quad z \in E,$$

that is

$$\frac{1}{z} \left( \frac{z/(1 - z)^2 - (1 + e^{i\theta})z/(1 - z)}{-e^{i\theta}} \right) \ast g(z) \neq 0, \quad \theta \in [0, 2\pi), \quad z \in E,$$

where $\ast$ denotes the Hadamard product of two functions. Since

$$-e^{i\theta} h_{\theta}(z) := \frac{z}{(1 - z)^2} - (1 + e^{i\theta}) \frac{z}{1 - z} = -e^{i\theta} + \sum_{n=2}^\infty (n - 1 - e^{i\theta}) z^n$$
where \(|n - 1 - e^{i\theta}| \leq n\) it is clear from the results in [1] that a sufficient condition in order that \(N_{\delta}(f) \subset T\) may hold for some function \(f\) in \(A\) is that

\[
\left| \frac{h_{\theta}(z) \ast f(z)}{z} \right| \geq \delta, \quad z \in E, \theta \in [0, 2\pi).
\]

Now let \(f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in \tilde{T}\). We have

\[
f(z) \ast h_{\theta}(z) = zf'(z) - (1 + e^{i\theta})f(z).
\]

with \(\text{Re} \{1 - e^{i\theta}zf''(z)/f'(z)\} \geq 1 - |zf''(z)/f'(z)| > 0\). This shows, since \(f \in \tilde{T} \subset C\), that the functions \(h_{\theta}(z) \ast f(z)\) are close-to-convex univalent. We also get the estimate

\[
|f(z) \ast h_{\theta}(z)| \geq \int_0^{|z|} e^{-\frac{r}{n}}(1 - u^n) \, du = |z| e^{-\frac{r}{n}}
\]

so that according to (3), \(N_{\delta}(f) \subset T\) for \(\delta_n = e^{-1/n}\). The sharpness of the result is seen from the function \(f(z) := \int_0^z e^{u^n} \, du;\) in fact \(g(z) := f(z) + \delta_n z^{n+1}/(n+1) \in N_{\delta}(f)\) and \(g'(z) = f'(z) + \delta_n z^n = 0\) if \(z^n = -1\). This completes the proof of Theorem 1.

**Proof of Theorem 2.** The proof of Theorem 2 is more direct. We first remark that from the definition of \(T\), we have

\[
g(z) \in T \Rightarrow g(z) = z \left( \frac{g_1(z)}{z} \right) \text{' } \text{for some function } g_1 \in T
\]

so that if \(g(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in T\), we get from (1) and Schwarz lemma that

\[
e^{-\frac{r}{n}} \leq \left| \frac{g(z)}{z} \right| \leq e^{\frac{r}{n}}.
\]

\[
\left| \frac{z g'(z)}{g(z)} \right| \leq r |z|^n.
\]

Now let \(0 < \theta < 2\pi;\) we have, according to (4) and (5), for \(z \in E\),

\[
\left| \frac{g(z) \ast h_{\theta}(z)}{z} \right| = \left| g'(z) - (1 + e^{i\theta}) \frac{g(z)}{z} \right| \geq \left| \frac{g(z)}{z} \right| \left( 1 - \frac{z g'(z)}{g(z)} - 1 \right) \geq e^{-\frac{r}{n}}(1 - r |z|^n)
\]

from which it follows, according to (3), that \(N_{\delta_n}(g) \subset T\) for \(\delta_n = (1 - r) e^{-r/n}\). The sharpness of the result is seen from the function \(g(z) := ze^{r\zeta^n}/\zeta^2;\) in fact,
\[ f(z) := g(z) + \delta_n z^{n+1}/(n + 1) \in N_\delta(g) \text{ and } f'(z) = 0 \text{ if } z^n = -1. \] This completes the proof of Theorem 2.

**Proof of Theorem 3.** Let \( h_\theta(z) \) be defined as before. Since
\[
\frac{f(z) \ast h_\theta(z)}{z} = -e^{i\theta} f(z) \left( \frac{f'(z)}{f(z)} - 1 \right) - e^{i\theta}
\]
it is clear from the hypothesis on \( D \), that, \(|f(z)/z|\) being bounded in \( E \),
\[
\inf \left| \frac{f(z) \ast h_\theta(z)}{z} \right| = 0
\]
where the inf is taken over all \( z \in E, \theta \in [0, 2\pi) \).

We now proceed to show Theorem 3 following an idea due to Ruscheweyh [1]. Let \( \delta > 0 \) and \( n \) some integer \( > 2 \). Choose a point \( z_0 \in E \) and \( \theta \in [0, 2\pi) \) such that for
\[
M := \frac{f(z) \ast h_\theta(z)}{z_0^n}
\]
we have
\[
M = f \ast h_\theta(z_0) < \delta \frac{n - 2}{n}
\]
This is always possible because of (6) and the fact that the function \( f(z) \bullet h_\theta(z) \), \( f \) being in the class \( T \), is nonvanishing for \( z \neq 0 \). We then define the function
\[
g(z) := f(z) - nz^n/a_n \text{ where } a_n := h_\theta^{(n)}(0)/n! = (n - 1 - e^{i\theta})/(-e^{i\theta})
\]
it is clear that \(|a_n| \geq n - 2\) so that \( n |\mu/a_n| < n |\mu|/(n - 2) < \delta \) and \( g \in N_\delta(f) \); but on the other side we have
\[
\frac{g \ast h_\theta(z_0)}{z_0} = \frac{f \ast h_\theta(z_0)}{z_0} - \mu z_0^{n-1} = 0
\]
which shows that \( g \notin T \). This completes the proof of Theorem 3.

**References**


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