SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

FOR ANY \( X \),
THE PRODUCT \( X \times Y \) IS HOMOGENEOUS FOR SOME \( Y \)

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Abstract. We prove that for every topological space \( X \) there exists a cardinal \( k \) and a nonempty subspace \( Y \subseteq X^k \) such that the product \( X \times Y \) is homogeneous. This answers a question of A. V. Arhangel'skií.

A topological space in which every point can be mapped to every other point by a homeomorphism of the space onto itself is called homogeneous. Answering a question of A. V. Arhangel'skií [A], Jan van Mill [VM] has constructed an example of a rigid (= no autohomeomorphisms beyond the identity) compact space \( X \) such that \( X \times X \) is homogeneous. It is not known whether for every compact space \( X \) there exists a nonempty compact space \( Y \) such that \( X \times Y \) is homogeneous (cf. [DV, Question 6.3]). We show if the requirement of compactness is omitted, such a \( Y \) always does exist.

Theorem. For every nonempty topological space \( X \) there exists a nonempty topological space \( Y \) such that \( X \times Y \) and \( Y \) are homeomorphic and homogeneous.

Proof. Let \( A \) be an infinite set of cardinality \( |A| \geq |X| \). In the cube \( X^A \), consisting of all functions \( f: A \to X \), consider the subspace \( Y = \{ f \in X^A : |f^{-1}(x)| = k \text{ for every } x \in X \} \). Clearly \( X \times Y \) and \( Y \) are homeomorphic. Let \( g \in Y \) and \( h \in Y \). Since \( |g^{-1}(x)| = |h^{-1}(x)| \) for each \( x \in X \), there exists a permutation \( p \) of the set \( A \) such that \( p(h^{-1}(x)) = g^{-1}(x) \) for each \( x \in X \), which means \( h = g \circ p \). The mapping \( f \mapsto f \circ p \) \( (f \in Y) \) is an autohomeomorphism of \( Y \) which maps \( g \) to \( h \). Hence \( Y \) is homogeneous, and so is \( X \times Y \).

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