SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

FOR ANY $X$, THE PRODUCT $X \times Y$ IS HOMOGENEOUS FOR SOME $Y$

VLADIMIR V. USPENSKIĬ

Abstract. We prove that for every topological space $X$ there exists a cardinal $k$ and a nonempty subspace $Y \subseteq X^k$ such that the product $X \times Y$ is homogeneous. This answers a question of A. V. Arhangel'skiĭ.

A topological space in which every point can be mapped to every other point by a homeomorphism of the space onto itself is called homogeneous. Answering a question of A. V. Arhangel'skiĭ [A], Jan van Mill [vM] has constructed an example of a rigid (= no autohomeomorphisms beyond the identity) compact space $X$ such that $X \times X$ is homogeneous. It is not known whether for every compact space $X$ there exists a nonempty compact space $Y$ such that $X \times Y$ is homogeneous (cf. [DvM, Question 6.3]). We show if the requirement of compactness is omitted, such a $Y$ always does exist.

Theorem. For every nonempty topological space $X$ there exists a nonempty topological space $Y$ such that $X \times Y$ and $Y$ are homeomorphic and homogeneous.

Proof. Let $A$ be an infinite set of cardinality $k \geq |X|$. In the cube $X^A$, consisting of all functions $f: A \to X$, consider the subspace $Y = \{f \in X^A: |f^{-1}(x)| = k \text{ for every } x \in X\}$. Clearly $X \times Y$ and $Y$ are homeomorphic. Let $g \in Y$ and $h \in Y$. Since $|g^{-1}(x)| = |h^{-1}(x)|$ for each $x \in X$, there exists a permutation $p$ of the set $A$ such that $p(h^{-1}(x)) = g^{-1}(x)$ for each $x \in X$, which means $h = g \circ p$. The mapping $f \mapsto f \circ p$ ($f \in Y$) is an autohomeomorphism of $Y$ which maps $g$ to $h$. Hence $Y$ is homogeneous, and so is $X \times Y$.

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REFERENCES


DEPARTMENT OF TOPOLOGY, FACULTY OF MECHANICS AND MATHEMATICS, MOSCOW STATE UNIVERSITY, MOSCOW 234, 117234, USSR