A CHARACTERIZATION OF UNCONDITIONED WEAK SEQUENCES

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Abstract. Unconditioned weak sequences are characterized in terms of the Koszul homology and applications to the theory of Buchsbaum rings and in a special case to the resolution of the ideal generated by the weak sequence are considered.

Introduction. In the theory of local rings there has been a rapid development of the theory of Buchsbaum rings, for basic facts see [8, 4, 5, 6]. Buchsbaum rings may be defined by the property that for every system of parameters \( \mathbf{r} \) of the local ring \(( R, m)\) the difference of length\( (R/(\mathbf{r}))\) and multiplicity of the ideal \((\mathbf{r})\) is an invariant, i.e. does not depend on the choice of the system of parameters \( \mathbf{r} \), or by the property: every system of parameters \( x_1, \ldots, x_n \) is a weak sequence, i.e.

\[ (\ast) \quad m((x_1, \ldots, x_{i-1}): x_i) \subseteq (x_1, \ldots, x_{i-1}) \text{ for all } 1 \leq i \leq n. \]

In the following we will say that a sequence of elements \( x_1, \ldots, x_n \) is an unconditioned weak sequence if it satisfies \((\ast)\) for any renumbering. Hence every system of parameters in a Buchsbaum ring is certainly an unconditioned weak sequence.

Another characterization of Buchsbaum rings follows from the result in [7], saying: for every system of parameters \( \mathbf{r} \) the Koszul homology \( H_a(\mathbf{r}) \) is a vector-space, equivalently \( H_i(\mathbf{r}) \) is a vectorspace. Hence it is equivalent demanding the weak sequence property for all systems of parameters \( \mathbf{r} \) of a local ring and demanding \( H_a(\mathbf{r}) \) \( (H_i(\mathbf{r}))\) is a vectorspace for all such \( \mathbf{r} \). So one may ask: given any sequence \( x_1, \ldots, x_n \) in a local ring, what does it mean for the Koszul homology if we demand \( \mathbf{r} \) to be an unconditioned weak sequence?

In this paper we want to answer this question. It turns out that the assumption for \( \mathbf{r} \) to be an unconditioned weak sequence does not imply in general that the Koszul homology modules \( H_i(\mathbf{r}) \) for \( i \geq 1 \) are vectorspaces, but that \( H_i(\mathbf{r}) \) modulo a certain submodule is a vectorspace. These submodules vanish e.g. if \( \mathbf{r} \) has the additional property to be unconditioned relatively regular in the sense of Fiorentini [1]. For a system of parameters \( \mathbf{r} \) of a Buchsbaum ring this property of course holds, since \( \mathbf{r} \) is even a (unconditioned) \( d \)-sequence (see [2]).

Further we show that for an unconditioned weak sequence \( \mathbf{r} \) the Koszul complex is always a (module-)direct summand of the minimal \( R \)-free resolution of \( R/(\mathbf{r}) \) for \( n \leq 2 \). We then show that if for every multi-index \((v_1, \ldots, v_n)\) the syzygetic part of
the Koszul homology \( \tilde{H}_i(x_1^n, \ldots, x_n^n; M) \) is a vectorspace for a finitely generated \( R \)-module \( M \), then \( x_1^n, \ldots, x_n^n \) is a weak sequence on \( M \) for all \( (n_1, \ldots, n_n) \). Hence from this fact and our characterization of weak sequences the characterization of Buchsbaum rings derived from Suzuki [7] follows and, more generally, together with the result in [7], this gives another proof of Shimoda’s characterization of Buchsbaum modules (cf. [3]).

Finally we consider, in the case that the \( H_i(\tau) \) are vectorspaces, the resolution of \( R/(x_1, x_2) \) as an \( R \)-module for an unconditioned weak sequence \( x_1, x_2 \) and give a condition when this resolution is obtained by the Koszul complex and copies of the resolutions of the residue class field suitably lifted.

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**Notations.** All rings \((R, m)\) will be commutative with 1, noetherian and local with maximal ideal \( m \). Let \( x_1, \ldots, x_n \) be elements of \( R \). We will write \( \tau \) for the whole sequence \( x_1, \ldots, x_n \); \( (\tau) \) denotes the \( R \)-ideal generated by \( \tau \). For any subset \( I \subseteq \{1, \ldots, n\} \) and any \( i \in I \) we write \( I_i \) for the subset of all elements of \( I \) excluding \( i \). Also \( x_1, \ldots, \hat{x}_i, \ldots, x_n \) stands for the sequence with \( x_i \) omitted.

Let \( I \subseteq \{1, \ldots, n\} \), then the Koszul complex \( K := K((x_i)_{i \in I}; M) \) for a \( R \)-module \( M \) is the \( R \)-complex whose underlying graded module is \( \bigwedge (\bigoplus_{i \in I} RT_i) \otimes_R M \) and the basis elements \( T_i \) map to \( x_i \) for \( i \in I \). If \( J = \{j_1, \ldots, j_s\} \subseteq I \) with \( j_1 < \cdots < j_s \), the element \( T_J \) denotes the product \( T_{j_1} \cdots T_{j_s} \in K((x_i)_{i \in I}; R) \). For our convenience we will also use the index \( J \) for the coefficient of the basis element \( T_J \). \( Z_k(K), B_k(K) \) denote the submodule of cycles and boundaries of \( K_k \) respectively. For the homology modules \( H_i(K(\tau; M)) \) we will also write \( H_i(\tau; M) \) for short.

**Main section.**

**Lemma 1.** Let \((R, m)\) be a local ring and \( x_1, \ldots, x_n \) a sequence of elements of \( R \) such that \( (x_1, \ldots, x_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n) \) for all \( 1 \leq i \leq n \); then \( Z_k(K(\tau; R)) \subseteq mK_k(\tau; R) \) for all \( k \geq 1 \).

**Proof.** Assume the assertion does not hold, so there is an \( I \subseteq \{1, \ldots, n\} \) such that
\[
T_I - \sum_{J \neq I} \lambda_J T_J \in Z(K) \quad (\lambda_J \in R),
\]
i.e.
\[
\delta(T_I) - \sum_{J \neq I} \lambda_J \delta(T_J) = 0,
\]
hence
\[
\sum_{i \in I} (-1)^i \cdots x_i T_i - \sum_{J \neq I} \lambda_J \sum_{j \in J} (-1)^i \cdots x_j T_j = 0.
\]
So comparing coefficients of the same basis elements we obtain the inclusion of ideals \((x_i)_{i \in I} \subseteq (x_j)_{j \in J} \), a contradiction to our assumption.

**Lemma 2.** Let \((R, m)\) be a local ring, \( x_1, x_2 \in R \) be an unconditioned weak sequence with \( (x_i) \subseteq (x_1, x_2) \) for \( i = 1, 2 \), then the Koszul complex \( K(x_1, x_2; R) \) is a (module-) direct summand of the minimal \( R \)-free resolution of \( R/(x_1, x_2) \).
Proof. As $K_0(x_1, x_2) = R$, $K_1$ is the free module $K_1 = RT_1 \oplus RT_2$ with basis elements $T_1, T_2$ which map to $x_1, x_2$ respectively, we only have to show that $x_2 T_1 - x_1 T_2 \notin m Z_t(K)$.

So assume $z = x_2 T_1 - x_1 T_2 \in m Z_t(K)$, then we have a representation $z = \sum \alpha_i z_i$, $\alpha_i \in m$, $z_i \in Z_t(K)$. We write $z_i = a_i T_1 - b_i T_2$, $a_i, b_i \in R$, then $\sum \alpha_i a_i = x_2$, $\sum \alpha_i b_i = x_1$. Since $z_i \in Z_t(K)$, we have $b_i \in (x_1 : x_2)$ for all $i$ and therefore $a_i b_i = \delta_i x_1$ for some $\delta_i \in R$, as $x_1, x_2$ form a weak sequence. Using $\sum \alpha_i b_i = x_1$, we obtain that there is an $i_0$ such that $\delta_{i_0} \notin m$, hence a unit in $R$. Consider $w := \delta_{i_0}^{-1} z_{i_0} := a T_1 - b T_2 \in Z_t(K)$, put $a := a_{i_0}$, then

$$aw - (x_2 T_1 - x_1 T_2) = (aa - x_2) T_1 \in m Z_t(K).$$

Now $w = a T_1 - b T_2$ is a cycle, so $a \in (x_2 : x_1)$, hence $aa \in (x_2)$. So we may write $aw - (x_2 T_1 - x_1 T_2) = \mu x_2 T_1$ with some $\mu \in R$. Therefore $\mu x_1 x_2 = 0$, and by the weak sequence property we derive $\mu m^2 = 0$.

If $x_1 \notin m^2$ or $x_2 \notin m^2$, then certainly $z = x_2 T_1 - x_1 T_2 \in m^2 K \supset m Z_t(K)$ a contradiction to our assumption. Hence we may assume that $x_1, x_2 \in m^2$, so $x_2 \mu \in m^2 \mu = 0$ and therefore $aw - (x_2 T_1 - x_1 T_2) = 0$. So $aa = x_2$ and $ab = x_1$.

As $b \in (x_1 : x_2)$, we have $mb \subseteq (x_1)$. Now consider the multiplication map $b \cdot : m \to (x_i)$ given by the multiplication with $b$. We have the exact sequence

$$0 \to \text{Ker}(b \cdot) \to m \to (x_i) \to 0.$$

Tensoring with $R/m$, we have a surjective map $m/m^2 \to (x_i)/m(x_i)$. Since $ab = x_1$, $a \in m \setminus m^2$. Let $m_1, \ldots, m_n$ denote a minimal system of generators of $m$ with $m_1 = a$ and $m_i b \in m(x_i)$ for $i = 2, \ldots, n$. Now write $m_i b = \gamma_i x_1$ with $\gamma_i \in m$ for $i = 2, \ldots, n$; then $a \gamma_i - m_i \in \text{Ker}(b \cdot)$, hence $a \gamma_i - m_i \in (0 : x_1) = (0 : m)$ for $i = 2, \ldots, n$.

So for all $i \geq 2$, $j \geq 1$ we have $m_j (a \gamma_i - m_i) = 0$, hence $m_i m_j \in m^3 \cap (a)$, therefore $m^2/m^3$ is generated by the class of $m_i^2 = a^2$, so $m^2$ is generated by $a^2$ by Nakayama’s Lemma, hence $m^k$ is generated by $a^k$ for all $k \geq 2$. So we may write $x_1 = u_1 a^{n_1}$, $x_2 = u_2 a^{n_2}$ with $u_1, u_2$ units in $R$ and $n_1, n_2$ natural numbers. Since $n_1 \leq n_2$ or $n_2 \leq n_1$ we have $(x_1, x_2) = (x_1)$ or $(x_1, x_2) = (x_2)$ the desired contradiction.

Proposition 1. Let $(R, m)$ be a local ring, $x_1, \ldots, x_n$ be an unconditioned weak sequence such that $(x_1, \ldots, x_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$, then the elements $d(T_i)$ for $I \subseteq \{1, \ldots, n\}$, $\# I \geq 2$ form part of a basis of $Z(K)/m Z(K)$, where $K = K(\chi, R)$.

Proof. Let $I \subseteq \{1, \ldots, n\}$ and denote

$$z = \sum_{i \in I} (-1)^{\sigma(i, I)} x_i T_i,$$

where $\sigma(i, I) = \# \{ j \in I / j < i \}$, then we have to show $z \notin m Z(K)$.

Let $R \to \overline{R} := R/(x_i)_{i \notin I}$ be the canonical surjection, which induces a homomorphism $K(x, R) \to K(\chi, \overline{R})$. We reduce $K(x, \overline{R})$ modulo the ideal $\langle T_i \rangle_{i \notin I}$ and obtain a homomorphism of differential graded algebras

$$K(\chi, \ldots, x_n, R) \to K((x_i)_{i \in I}; R/\langle x_i \rangle_{i \notin I}).$$
Now it is enough to show that the image of $z$ under this homomorphism is not contained in $\overline{m}K((x_i)_{i \in I}; R/(x_j)_{j \notin I})$, where $\overline{m}$ denotes the maximal ideal of $R/(x_j)_{j \notin I}$.

So without loss of generality by renumbering we only have to show

$$z = \sum \frac{1}{i} \cdot T_i \cdot \cdot \cdot T_n \in m \subseteq Z_{n-1}(K).$$

Let us assume this is not true and reduce again modulo $x_3, \ldots, x_n$, then

$$z = x_3 T_2 \cdot \cdot \cdot T_n - x_2 T_3 \cdot \cdot \cdot T_n \in m \subseteq Z_{n-1}(K(x_3, R/(x_3, \ldots, x_n))),$$

hence there exist $a_j \in m$, $z_j \in Z_{n-1}(K(x_3, R/(x_3, \ldots, x_n)))$, $z_j = \sum (-1)^{i+1} a_j T_{i+1} \cdot \cdot \cdot T_n$, such that $z = \sum a_j z_j$. Since $\delta z_j = 0$, we have $a_j x_2 T_3 \cdot \cdot \cdot T_n - a_j x_3 T_2 \cdot \cdot \cdot T_n = 0$, hence $a_j x_2 - a_j x_3 = 0$ for all $j$ and $\sum a_j a_j = \overline{x}_1$, $\sum a_j a_j = \overline{x}_2$. So consider $w_j := a_j T_1 - a_j T_2 \in Z_1(K(x_1, \overline{x}_2; R/(x_3, \ldots, x_n)))$, then

$$\sum a_j w_j = x_2 T_1 - x_1 T_2 \in m Z_1(K(x_1, \overline{x}_2; R/(x_3, \ldots, x_n))),$$

which contradicts Lemma 2. This proves the assertion of the Proposition.

**Remark.** There exist unconditioned weak sequences $z$, for which the assumption $(x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$ does not hold and hence the assertion of the proposition cannot be true in these cases.

**Example.** Let $(R, m)$ be a regular local ring of dimension 1 and let $x$ be a generator of $m$. Consider $x_1 := x$, $x_2 := x^r$ ($r \geq 1$). Then $(0 : x_1) = (0 : x_2) = 0$, $(x_1 : x_2) = R$, hence $m(x_1 : x_2) = (x_1)$, $(x_2 : x_1) = (x^r-1)$, hence $m(x_2 : x_1) = (x_2)$.

**Lemma 3.** Let $(R, m)$ be a local ring, $x_1, \ldots, x_n$ an unconditioned weak sequence such that $(x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$. Let $I \subseteq \{1, \ldots, n\}$ and denote $z = \sum_{i \in I} \alpha_i T_i \in K_1(x, \overline{R})$ a cycle, then for every $i \in I$ and for every $\beta \in m$ we have $\beta \alpha_i \in m(x_2)$.

**Proof.** By a suitable renumbering we may assume $I = \{1, \ldots, s\}$. Modulo the ideal $(x_3, \ldots, x_n)$ we consider $\overline{z} = \overline{a_1 T_1 + a_2 T_2}$. $\overline{z}$ is a cycle in $K_1(x, \overline{R})$. Now it is enough to show that $\beta \alpha_i \in m(x_2)$ and $\beta \alpha_2 \in m(x_1)$ modulo $(x_3, \ldots, x_n)$ for every $\beta \in m$. So without loss of generality $n = 2$, $z = a T_1 - b T_2$ and we show $\beta a \in m(x_2)$, $\beta b \in m(x_1)$ for $\beta \in m$.

So take any $\beta \in m$, then there exist $u_1, u_2 \in R$ such that $\beta a = u_2 x_2$, $\beta b = u_1 x_1$ by the weak sequence property.

Now $u_1, u_2$ cannot simultaneously be units in $R$. If so, we have $u_1 x_1 x_2 = u_2 x_1 x_2$, since $z = a T_1 - b T_2$ is a cycle, and hence by the weak sequence property $(u_1 - u_2) x_1 x_2 = 0$. So $a = u_2^{-1} u_1 x_1 x_2 \alpha$ for every $\alpha \in m^2$. Now since $x_1, x_2$ is an unconditioned weak sequence, so is $x_1, x_2$, where we denote $x'_1 := u_1 x_1$, $x'_2 := u_2 x_2$. Consider the cycle $z' := u_2 a T_1 - u_1 b T_2 \in K(x'_1, x'_2, R)$. Then

$$u_2^{-1} \beta z' = \beta a T'_1 - u_2^{-1} u_1 \beta b T'_2 = \beta a T'_1 - \beta b T'_2$$

$$= x'_2 T'_1 - x'_1 T'_2 \in m Z(K),$$

which contradicts the assertion of Lemma 2.

So at least one of the coefficients $u_1, u_2$ is a nonunit. Let $u_1 \in m$. We have to show that also $u_2 \in m$. So assume $u_2$ is a unit, without loss of generality $u_2 = 1$. Then
$x_1x_2 = \beta ax_1 = \beta b x_2 = u_1 x_1 x_2$, hence $x_1 x_2 = 0$. We obtain $x_2 \in (0 : x_1) = (0 : m)$, therefore $x_2 \in 0$ hence $(0 : x_2) = m$. But then we have $x_2 = \beta a \in m^2 = m(0 : x_2) = m(0 : m) = 0$ a contradiction.

**Corollary.** Let $(R, m)$ be a local ring, $x_1, \ldots, x_n$ an unconditioned weak sequence such that $(x_1, \ldots, \hat{x}_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$, then $m((x_j)_{j \in I_0} : x_i) \subseteq m(x_j)_{j \in I_0}$.

**Proof.** Let $\alpha \in (x_j)_{j \in I_0} : x_i$, $\beta \in m$, then there exist $y_j \in R$ for $j \in I$ such that $\alpha x_i = \sum_{j \in I_0} y_j x_j$. Then $\alpha T_i - \sum_{j \in I_0} y_j T_j$ is a cycle in $K(\mathbb{R}, R)$, hence by Lemma 3 we have $\beta \alpha \in m(x_j)_{j \in I_0}$.

We are now able to prove the following

**Theorem.** Let $(R, m)$ be a local ring, $x_1, \ldots, x_n$ be a sequence of elements such that $(x_1, \ldots, \hat{x}_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$, then the following are equivalent:

1. $(x_1, \ldots, x_n)$ is an unconditioned weak sequence.
2. For all $I \subseteq \{1, \ldots, n\}$, $1 \leq k \leq \#I$, $J_0 \subseteq I$ with $\#J_0 = \#I - k$:

$$H_k(K(x_i)_{i \in I})/i\left(\left([m(\mathbb{R})] \sum_{j \in I} K_k(x_j)_{j \in J} \right) \cap Z_k(K(x_i)_{i \in I})\right)$$

is a vectorspace,

3. For all $I \subseteq \{1, \ldots, n\}$, $i_0 \in I$:

$$H_1(K(x_i)_{i \in I})/i\left((m(\mathbb{R})K_1(x_i)_{i \in I_0}) \cap Z(K(x_i)_{i \in I})\right)$$

is a vectorspace,

4. For all $I \subseteq \{1, \ldots, n\}$, $i_0 \in I$:

$$H_1(K(x_i)_{i \in I})/H_1(K(x_i)_{i \in I_0})$$

is a vectorspace.

($i$ denotes the canonical map $Z_*(K) \to H_*(K)$.)

**Proof.** (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1). Consider the long exact homology sequence associated to $0 \to K(x_i)_{i \in I_0} \to K(x_i)_{i \in I} \to K(x_i)_{i \in I_0}[-1] \to 0$, then we obtain that the vector space $H_1(K(x_i)_{i \in I})/H_1(K(x_i)_{i \in I_0})$ is just the kernel of the multiplication map

$$R/ K(x_i)_{i \in I_0} \to R/ K(x_i)_{i \in I_0},$$

hence $m((x_i)_{i \in I_0} : x_i) \subseteq (x_i)_{i \in I_0}$.

1. $\Rightarrow$ (2). Let $\#I = r$, $1 \leq k \leq r$ and $J_0 \subseteq I$ with $\#J_0 = r - k$. Consider

$$z = \sum_{\#J_0 = k} \alpha \tau T \in Z_k(K(x_i)_{i \in I}).$$
Denote $s_0 := I \setminus J_0$, then $\alpha_{s_0} \in ((x_j)_{j \in \mathbb{N}_0} : (x_j')_{j \in \mathbb{N}_0})$, so for any $\beta \in m$: $\beta \alpha_{s_0} \in m(x_j)_{j \in \mathbb{N}_0}$ by the corollary of Lemma 3. So write

$$\beta \alpha_{s_0} = \sum_{j \not\in s_0} \gamma_j x_j \text{ with } \gamma_j \in m,$$

hence

$$\beta z - \delta \left( \sum_{j \not\in s_0} \gamma_j T_j T_{s_0} \right) \in \left( m(\tau) \sum_{j \not\in s_0} K_k(x_j),_{\tau} \right) \cap Z_k(K(x_i),_{\tau}),$$

since for each $s_0$ we also have $\beta \alpha_{s_0} \in m(\tau)$ again by the corollary of Lemma 3.

**Remark 1.** If $x_1, \ldots, x_n$ is an unconditioned relatively $m$ regular sequence with respect to $m(\tau)$ in the sense of Fiorentini [1], then $m(\tau)K_k(x_i),_{\tau} \cap Z_k(K(x_i),_{\tau}) \subseteq mB_k(K(x_i),_{\tau})$ for all $I \subseteq \{1, \ldots, n\}$ by Fiorentini's result, hence the submodules by which we have to divide $H_k(K(x_i),_{\tau})$ in (2) and (3) in the theorem vanish.

For a system of parameters $\tau$ of a Buchsbaum ring this assumption certainly holds, since in this case $\tau$ is even a $\beta$-sequence (cf. [2]).

**Remark 2.** In general for unconditioned weak sequences the homology modules are not vectorspaces.

**Example.** Let $A = k[[x_1, x_2, y_1, y_2, z]]$ be the powerseries ring in five variables over a field $k$ and denote $m$ the maximal ideal of $A$.

Let

$$R := A/(y_1^2, y_1 y_2, y_2^2, y_1 x_2 - y_2 x_1, y_1 m^2, y_2 m^2, z^2, z y_1, z y_2 - y_1 x_2),$$

denote $m$ the maximal ideal of $R$ and $x_1, x_2, y_1, y_2, z$ the residues of the corresponding elements of $A$. Then

(a) $x_1, x_2$ is an unconditioned weak sequence.

(b) $H_1(x_1, x_2; R)$ is not a vectorspace.

**Proof.** (a) $(0 : x_1) = (0 : x_2) = (y_1, y_2, z)m,$

$$(x_1 : x_2) = (y_1, y_2, z)m + (y_1, x_1),$$

$$(x_2 : x_1) = (y_1, y_2, z)m + (y_2, x_2).$$

Obviously $x_1, x_2$ is an unconditioned weak sequence.

(b) Consider $y_2 T_1 - y_1 T_2 \in Z(K(x_1, x_2; R)).$ Then $0 \neq z y_2 T_1 - z y_1 T_2 = z y_2 T_1 \in mZ(K)$. Notice that $R$ is a homogeneous ring. Assume now that $z y_2 T_1 \in B(K)$, then there exists a linear form $f$ such that $\delta(f T_1 T_2) = z y_2 T_1$, but $\delta(f T_1 T_2) = f x_1 T_2 - f x_2 T_1$, hence $f \in (0 : x_1) \subseteq m^2$ a contradiction.

**Remark 3.** The example in the remark to Proposition 1 shows for $r \geq 3$ that the theorem is not valid without the assumption $(x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$. In this particular case $H_1(K(x_1)) = 0$ and $x^{-1}H_1(K(x_1, x_2)) \neq 0$.

Using the theorem and the following Proposition 2 we now would like to give a new proof of the characterization of Buchsbaum rings in terms of the Koszulhomology. This characterization is already known using Suzuki [7] and Shimoda [3]:

The following are equivalent:

(1) $R$ is a Buchsbaum ring.

(2) For all systems of parameters $\tau$: $H_*(\tau, R)$ is a vectorspace,
(3) for all systems of parameters $\mathfrak{r}: H_1(\mathfrak{r}, R)$ is a vectorspace,
(4) for all systems of parameters $\mathfrak{r}: \hat{H}_1(\mathfrak{r}, R)$ is a vectorspace,
(5) for all systems of parameters $\mathfrak{r}: H_1(\mathfrak{r}, R)$ is a vectorspace.

$(\hat{H}_1(\mathfrak{r}, M) := \text{coker}(H_1(\mathfrak{r}, (\mathfrak{r})M) \to H_1(\mathfrak{r}, M))$ for a $R$-module $M$.)

$(1) \Rightarrow (2)$ follows from our theorem and Remark 1. Obviously $(2) \Rightarrow (3) \Rightarrow (5)$ and
$(2) \Rightarrow (4) \Rightarrow (5)$. We only need to show $(5) \Rightarrow (1)$, hence e.g. every system of parameters $\mathfrak{r}$ is a weak sequence. The following proposition which we prove more general for the module case (cf. Introduction) implies the desired result.

**Proposition 2.** Let $(R, m)$ be a local ring, $M$ a finitely generated $R$-module and $x_1, \ldots, x_n$ a sequence of elements in $m$ such that $H_1(x_1^\ast, \ldots, x_n^\ast; M)$ is a vectorspace for all multi-indices $(v_1, \ldots, v_n)$ with $v_1 \geq 1$, then $x_1^\ast, \ldots, x_n^\ast$ is a weak sequence on $M$ for all $(v_1, \ldots, v_n)$.

**Proof.** It is only left to show that $\mathfrak{r}$ is a weak sequence. We will do this in two steps: (1) $m((x_1, \ldots, x_{n-1})M : x_n) \subseteq (x_1, \ldots, x_{n-1})M$. (2) If $H_1(x_1^\ast, \ldots, x_n^\ast; M)$ is a vectorspace for all $v_1, \ldots, v_n$, then $H_1(x_1^\ast, \ldots, x_{n-1}^\ast; M)$ is a vectorspace for all $v_1, \ldots, v_n$.

Then descending step by step we obtain the conclusion.

To (1). We have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & K(x_1, \ldots, x_{n-1}; (\mathfrak{r})M) & \to & K(x_1, \ldots, x_n; (\mathfrak{r})M) & \to & K(x_1, \ldots, x_{n-1}; (\mathfrak{r})M)[-1] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \to & K(x_1, \ldots, x_{n-1}; M) & \to & K(x_1, \ldots, x_n; M) & \to & K(x_1, \ldots, x_{n-1}; M)[-1] & \to & 0 \\
\end{array}
\]

Taking homology we get the following commutative diagram:

\[
\begin{array}{ccccccccc}
H_1(x_1, \ldots, x_n; (\mathfrak{r})M) & \xrightarrow{\varphi_1} & H_0(x_1, \ldots, x_{n-1}; (\mathfrak{r})M) & \to & \text{coker } \varphi_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
H_1(x_1, \ldots, x_n; M) & \xrightarrow{\varphi} & H_0(x_1, \ldots, x_{n-1}; M) & \to & \text{coker } \varphi & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
\hat{H}_1(x_1, \ldots, x_n; M) & \to & \text{coker } \psi & \to & \text{coker } \psi_0
\end{array}
\]

In other terms

\[
\begin{array}{ccccccccc}
H_1((\mathfrak{r}; (\mathfrak{r})M) & \to & (\mathfrak{r})M/(\mathfrak{r})(x_1, \ldots, x_{n-1})M & \xrightarrow{\varphi_1} & (\mathfrak{r})M/(\mathfrak{r})(x_1, \ldots, x_{n-1})M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
H_1(\mathfrak{r}, M) & \to & M/(x_1, \ldots, x_{n-1})M & \xrightarrow{\varphi} & (\mathfrak{r})M/(\mathfrak{r})(x_1, \ldots, x_{n-1})M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
\hat{H}_1(\mathfrak{r}; M) & \to & M/(\mathfrak{r}M) & \xrightarrow{\varphi_1} & (\mathfrak{r})M/(\mathfrak{r})(x_1, \ldots, x_{n-1}, x_n^2)M
\end{array}
\]

By the snake lemma $\hat{H}_1(\mathfrak{r}, M)$ maps onto the kernel of

\[
M/(\mathfrak{r}M) \xrightarrow{\psi_0} (\mathfrak{r})M/(x_1, \ldots, x_{n-1}, x_n^2)M,
\]
hence
\[ m\left( (x_1, \ldots, x_{n-1}, x_n^2)M : x_n \right) \subseteq (x_1, \ldots, x_n)M. \]
Now let \( \alpha \in (x_1, \ldots, x_{n-1})M : x_n \) we have to show that \( \alpha m \subseteq (x_1, \ldots, x_{n-1})M. \) But
\[ \alpha x_n \in (x_1, \ldots, x_{n-1})M \subseteq (x_1, \ldots, x_{n-1}, x_{n+1}^2)M \text{ for all } k \geq 1, \]
so \( \alpha x_n^k \in (x_1, \ldots, x_{n-1}, x_n^{2k})M. \) Now repeating the above argument for \( \alpha m \subseteq (x_1, \ldots, x_{n-1}, x_n^k)M \) for all \( k \geq 1, \) hence \( \alpha m \subseteq (x_1, \ldots, x_{n-1})M. \)

To (2). Again taking homology from \((+)\) we derive Diagram A.

By the snake lemma we have an exact sequence
\[ K \rightarrow C(x_1, \ldots, x_n; M) \rightarrow \tilde{H}_1(\mathbb{Z}; M). \]

The map \( K \rightarrow C(x_1, \ldots, x_n; M) \) is actually the zero map. To prove this, consider \( \alpha \in M \) with \( \bar{\alpha} \in K, \) then \( \alpha \in (x_1, \ldots, x_{n-1})M. \) Let \( \alpha = \sum_{i=1}^{n-1} x_i \beta_i, \beta_i \in M \) then
\[ x_n \alpha = \sum_{i=1}^{n-1} x_n x_i \beta_i, \]
so \( z = \alpha T_n - \sum_{i=1}^{n-1} x_i T_i \) is a cycle in \( K(\mathbb{Z}, (\mathbb{Z})M) \) which maps onto \( \bar{\alpha}. \) But \( z = \delta(\sum_{i=1}^{n-1} \beta_i T_i T_n) \) is a boundary in \( K_1(x_1, \ldots, x_n; M). \) So \( C(x_1, \ldots, x_n; M) \) is a vector-space since \( \tilde{H}_1(\mathbb{Z}; M) \) is.

Again the same argument is true for the sequence \( x_1, \ldots, x_{n-1}, x_n^k \) for all \( k \geq 1, \) hence for all \( k \geq 1: C(x_1, \ldots, x_{n-1}, x_n^k, M) \) is a vector-space. Therefore also
\[ C'(x_1, \ldots, x_{n-1}, x_n^k, M) \]
\[ := \text{coker}(H_1(x_1, \ldots, x_{n-1}; (x_1, \ldots, x_{n-1}, x_n^k)M) \rightarrow H_1(x_1, \ldots, x_{n-1}; M)) \]
is a vector-space.

Finally consider Diagram B.

By the snake lemma we have the exact sequence
\[ L \rightarrow \tilde{H}_1(x_1, \ldots, x_{n-1}; M) \rightarrow C'(x_1, \ldots, x_{n-1}, x_n^k; M). \]

So \( \tilde{H}_1(x_1, \ldots, x_{n-1}; M) \) modulo the submodule generated by all cycles \( z = \sum_{i=1}^{n-1} \alpha_i T_i \) with \( \alpha_i \in (x_1, \ldots, x_{n-1}, x_n^k)M \) is a vector-space, hence for every cycle \( z \in K_1, \) we have
\[ \mathfrak{m} z \subseteq \bigcap_{k \geq 1} (x_1, \ldots, x_{n-1}, x_n^k)K_1 = (x_1, \ldots, x_{n-1})K_1, \]
so \( \mathfrak{m} \tilde{H}_1(x_1, \ldots, x_{n-1}; M) = 0. \)

We now want to look at the minimal \( R \)-free resolution of \( R/(\mathfrak{z}) \) for an unconditioned weak sequence \( \mathfrak{z}. \)

By Lemma 2 the Koszul-complex is always a (module-)direct summand of this resolution for \( n \leq 2. \) But beside this fact in general very little can be said about what this resolution looks like. If \( \mathfrak{z} \) is just a sequence of one element \( x_1, \) then the resolution of \( R/(x_1) \) is given by
\[ \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \xrightarrow{\varphi} R \rightarrow R \rightarrow R/(x_1) \rightarrow 0, \]
where \( (F_\ast, d_\ast) \) is a direct sum of \( \dim_R/\mathfrak{m}(0 : \mathfrak{m}) = \dim_R/\mathfrak{m} H_i(x_1) \) copies of the minimal \( R \)-resolution of the residue class field \( R/\mathfrak{m} \) and \( \varphi \) maps the baseelements of \( F_0 \) to representatives of a minimal system of generators of \( H_i(x_1). \) So in this case finding the \( R \)-resolution of \( R/(x_1) \) is equivalent to finding the \( R \)-resolution of \( R/\mathfrak{m}. \)
0 \rightarrow H_i(x_1, \ldots, x_{n-1}; (\xi)M) / x_n H_i(\ldots) \rightarrow H_i(\xi; (\xi)M) \rightarrow (\xi)(x_1, \ldots, x_{n-1})M : x_n/(\xi)(x_1, \ldots, x_{n-1})M \rightarrow 0

0 \rightarrow H_i(x_1, \ldots, x_{n-1}; M) / x_n H_i(\ldots) \rightarrow H_i(\xi; M) \rightarrow (x_1, \ldots, x_{n-1})M : x_n/(x_1, \ldots, x_{n-1})M \rightarrow 0

\downarrow
\downarrow
C(x_1, \ldots, x_n; M) \rightarrow \tilde{H}_i(\xi; M)
\downarrow
\downarrow
0 \rightarrow 0

\text{Diagram A}

H_i(x_1, \ldots, x_{n-1}; (x_1, \ldots, x_{n-1})M) \rightarrow H_i(x_1, \ldots, x_{n-1}; (x_1, \ldots, x_{n-1}, x_i^k)M) \rightarrow L \rightarrow 0

\downarrow
\downarrow
0 \rightarrow H_i(x_1, \ldots, x_{n-1}; M) \rightarrow H_i(x_1, \ldots, x_{n-1}; M) \rightarrow 0

\downarrow
\downarrow
\tilde{H}_i(x_1, \ldots, x_{n-1}; M) \rightarrow C'(x_1, \ldots, x_{n-1}, x_i^k; M)

\text{Diagram B}
So one may ask in general: Assume we know the $R$-resolutions of the homology modules $H_i(\tau)$, what can be said about the $R$-resolution of $R/(\tau)$ in terms of these given resolutions?

We want to give an answer to this question in a very special case: namely $\tau = x_1, x_2$ is an unconditioned weak sequence of two elements and $\tau$ is unconditioned $m$-relatively regular with respect to $m(\tau)$.

In this case $H_1(\tau), H_2(\tau)$ are vector-spaces by Remark 1 to the theorem. Denote $M, N$ free $R$-modules of rank $\dim H_1(\tau)$ and $\dim H_2(\tau)$ respectively and denote $(G, d)$ the resolution of the residue class field $R/m$. Below we will give an equivalent condition for the $R$-resolution of $R/(\tau)$ to be isomorphic to

\[(\times) \quad K \oplus (G \otimes M[-2]) \oplus (G \otimes N[-3])\]

with obvious maps $G_0 \otimes M[-2] \to K_1$, $G_0 \otimes N[-3] \to K_2$ killing the homology $H_*(K)$.

Denote $(z_j | j = 1, \ldots, s)$ a set of cycles in $K_1(\tau)$ which represent a $R/m$ vector-space basis of $H_1(\tau)$. Let $m_1, \ldots, m_n$ be a minimal system of generators of the maximal ideal $m$. For every pair $m_i, z_j$ we choose an element $\gamma(m_i, z_j) \in m$ such that $\delta(\gamma(m_i, z_j)T_1T_2) = m_iz_j$. This is possible, see Remark 1. Let $\mathfrak{A}$ be the $R$-ideal

\[\mathfrak{A} := \left( \sum_i r_i \gamma(m_i, z_j) / j = 1, \ldots, s; \sum_i r_i m_i = 0 \right).\]

Notice that the choices for $\gamma(m_i, z_j)$ are determined up to socle-elements, so as any relation $\sum r_i m_i = 0$ only involves $r_i \in m$ the ideal $\mathfrak{A}$ is well defined.

If $\sum r_i m_i = 0$ then

\[\sum_i r_i \gamma(m_i, z_j)(x_1 T_1 - x_2 T_2) = \delta(\sum_i r_i \gamma(m_i, z_j) T_1 T_2) = \left( \sum_i r_i m_i \right) z_j = 0.\]

Hence $\sum r_i \gamma(m_i, z_j) \in (0 : m)$, so we obtain $\mathfrak{A} \subseteq (0 : m)$.

Claim. The minimal $R$-resolution $(F, d)$ of $R/(\tau)$ is isomorphic to $(\times)$ if and only if $\mathfrak{A} = 0$.

Proof. Clearly the $R$-resolution $F$ of $R/(\tau)$ starts with

\[F_0 = R,\]
\[F_1 = R T_1 \oplus R T_2; T_1 \mapsto x_1, T_2 \mapsto x_2,\]
\[F_2 = R T_1 T_2 \oplus G_0 \otimes M[2].\]

Denote $S_j, j = 1, \ldots, s$, a free basis of $M[-2]$ and let $1 \otimes S_j$ map to $z_j$ for $j = 1, \ldots, s$.

So we may define for $i = 1, \ldots, n, j = 1, \ldots, s$:

\[d(V_i \otimes S_j) = m_i(1 \otimes S_j) - \gamma(m_i, z_j)T_1T_2,\]

where $V_1, \ldots, V_n$ denote a $R$-basis of $G_1$ which is mapped to $m_1, \ldots, m_n$ in $G$.

If $\mathfrak{A} = 0$, then we can define

\[d_{|G \otimes M[-2]) \otimes M[-2]} = (d_G)_{|G \otimes M[-2]} \otimes \text{id}_{|M[-2]},\]
\[d_{|G \otimes N[-3]) \otimes M[-2]} = (d_G)_{|G \otimes N[-3]} \otimes \text{id}_{|N[-3]}\]

and $G_0 \otimes N[-3] \to K_2$ the obvious map killing the homology $H_2(K)$. So we have the resolution as desired.
If $\mathfrak{A} \neq 0$, then there is a relation $\sum r_j m_j = 0$, $j \in \{1, \ldots, s\}$ with $0 \neq \alpha_j = \sum r_j \gamma(m_j, z_j) \in (0 : m)$. Hence the element $\alpha_j T_1 T_2$ is already killed by $d(G_1 \otimes M[-2])$ and therefore must not be killed by a basis element of $G_0 \otimes N[-3]$. Also $(\sum r_j m_j) \cdot (1 \otimes S_j) = 0$, but $(\sum r_j V_j) \otimes S_j$ is not a cycle in $F$ although $\sum r_j V_j$ is a cycle in $G$. So in this case $(\times)$ even is not a complex. Hence the $R$-resolution of $R/(\mathfrak{A})$ is different from $(\times)$.

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