A CHARACTERIZATION OF UNCONDITIONED WEAK SEQUENCES

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ABSTRACT. Unconditioned weak sequences are characterized in terms of the Koszul homology and applications to the theory of Buchsbaum rings and in a special case to the resolution of the ideal generated by the weak sequence are considered.

Introduction. In the theory of local rings there has been a rapid development of the theory of Buchsbaum rings, for basic facts see [8, 4, 5, 6]. Buchsbaum rings may be defined by the property that for every system of parameters \( \tau \) of the local ring \((R, m)\) the difference of length\( (R/(\tau)) \) and multiplicity of the ideal \((\tau)\) is an invariant, i.e. does not depend on the choice of the system of parameters \( \tau \), or by the property: every system of parameters \( x_1, \ldots, x_n \) is a weak sequence, i.e.

\[
(*) \quad m((x_1, \ldots, x_{i-1}): x_i) \subseteq (x_1, \ldots, x_{i-1}) \quad \text{for all } 1 \leq i \leq n.
\]

In the following we will say that a sequence of elements \( x_1, \ldots, x_n \) is an unconditioned weak sequence if it satisfies \((*)\) for any renumbering. Hence every system of parameters in a Buchsbaum ring is certainly an unconditioned weak sequence.

Another characterization of Buchsbaum rings follows from the result in [7], saying: for every system of parameters \( \tau \) the Koszul homology \( H_\bullet(\tau) \) is a vector space, equivalently \( H_1(\tau) \) is a vectorspace. Hence it is equivalent demanding the weak sequence property for all systems of parameters \( \tau \) of a local ring and demanding \( H_\bullet(\tau) \) \((H_1(\tau))\) is a vectorspace for all such \( \tau \). So one may ask: given any sequence \( x_1, \ldots, x_n \) in a local ring, what does it mean for the Koszul homology if we demand \( \tau \) to be an unconditioned weak sequence?

In this paper we want to answer this question. It turns out that the assumption for \( \tau \) to be an unconditioned weak sequence does not imply in general that the Koszul homology modules \( H_i(\tau) \) for \( i \geq 1 \) are vectorspaces, but that \( H_i(\tau) \) modulo a certain submodule is a vectorspace. These submodules vanish e.g. if \( \tau \) has the additional property to be unconditioned relatively regular in the sense of Fiorentini [1]. For a system of parameters \( \tau \) of a Buchsbaum ring this property of course holds, since \( \tau \) is even a (unconditioned) \( d \)-sequence (see [2]).

Further we show that for an unconditioned weak sequence \( \tau \) the Koszul complex is always a (module-)direct summand of the minimal \( R \)-free resolution of \( R/(\tau) \) for \( n \leq 2 \). We then show that if for every multi-index \((\nu_1, \ldots, \nu_n)\) the syzygetic part of
the Koszulhomology $\tilde{H}_i(x^n_1, \ldots, x^n_n; M)$ is a vectorspace for a finitely generated $R$-module $M$, then $x^n_1, \ldots, x^n_n$ is a weak sequence on $M$ for all $(\nu_1, \ldots, \nu_n)$. Hence from this fact and our characterization of weak sequences the characterization of Buchsbaum rings derived from Suzuki [7] follows and, more generally, together with the result in [7], this gives another proof of Shimoda's characterization of Buchsbaum modules (cf. [3]).

Finally we consider, in the case that the $H_i(\tau)$ are vectorspaces, the resolution of $R/(x_1, x_2)$ as an $R$-module for an unconditioned weak sequence $x_1, x_2$ and give a condition when this resolution is obtained by the Koszulcomplex and copies of the resolutions of the residue class field suitably lifted.

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**Notations.** All rings $(R, m)$ will be commutative with 1, noetherian and local with maximal ideal $m$. Let $x_1, \ldots, x_n$ be elements of $R$. We will write $\tau$ for the whole sequence $x_1, \ldots, x_n$; $(\tau)$ denotes the $R$-ideal generated by $\tau$. For any subset $I \subseteq \{1, \ldots, n\}$ and any $i \in I$ we write $I_i$ for the subset of all elements of $I$ excluding $i$. Also $x_1, \ldots, x_i, \ldots, x_n$ stands for the sequence with $x_i$ omitted.

Let $I \subseteq \{1, \ldots, n\}$, then the Koszulcomplex $K := K((x_i), i \in I; M)$ for a $R$-module $M$ is the $R$-complex whose underlying graded module is $\bigoplus_{i \in I} RT_i \otimes_R M$ and the basiselements $T_i$ map to $x_i$ for $i \in I$. If $J = \{j_1, \ldots, j_s\} \subseteq I$ with $j_1 < \cdots < j_s$, the element $T_J$ denotes the product $T_{j_1} \cdots T_{j_s} \in K((x_i), i \in J; R)$. For our convenience we will also use the index $J$ for the coefficient of the basiselement $T_J$. $Z_k(K), B_k(K)$ denote the submodule of cycles and boundaries of $K_k$ respectively. For the homology modules $H_i(K(\tau; M))$ we will also write $H_i(\tau; M)$ for short.

**Main section.**

**Lemma 1.** Let $(R, m)$ be a local ring and $x_1, \ldots, x_n$ a sequence of elements of $R$ such that $(x_1, \ldots, x_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$; then $Z_k(K(\tau, R)) \subseteq m K_k(\tau, R)$ for all $k \geq 1$.

**Proof.** Assume the assertion does not hold, so there is an $I \subseteq \{1, \ldots, n\}$ such that

$$T_I - \sum_{J \neq I} \lambda_J T_J \in Z(K) \quad (\lambda_J \in R),$$

i.e.

$$\delta(T_I) - \sum_{J \neq I} \lambda_J \delta(T_J) = 0,$$

hence

$$\sum_{i \in I} (-1)^i \cdot x_i T_i - \sum_{J \neq I} \lambda_J \sum_{j \in J} (-1)^{j} \cdot x_j T_j = 0.$$

So comparing coefficients of the same basiselements we obtain the inclusion of ideals $(x_i), i \in I \subseteq (x_j), j \in J$, a contradiction to our assumption.

**Lemma 2.** Let $(R, m)$ be a local ring, $x_1, x_2 \in R$ be an unconditioned weak sequence with $(x_i) \subseteq (x_1, x_2)$ for $i = 1, 2$, then the Koszulcomplex $K(x_1, x_2; R)$ is a (module-) direct summand of the minimal $R$-free resolution of $R/(x_1, x_2)$.
Proof. As $K_0(x_1, x_2) = R$, $K_1$ is the free module $K_1 = RT_1 \oplus RT_2$ with basis elements $T_1, T_2$ which map to $x_1, x_2$ respectively, we only have to show that $x_2T_1 - x_1T_2 \in \mathfrak{m}Z_i(K)$.

So assume $z = x_2T_1 - x_1T_2 \in \mathfrak{m}Z_i(K)$, then we have a representation $z = \Sigma a_i\cdot x_i$, $a_i \in \mathfrak{m}$, $x_i \in Z_i(K)$. We write $z_i = a_iT_1 - b_iT_2$, $a_i, b_i \in R$, then $\Sigma a_i\cdot x_i = x_2, \Sigma a_i\cdot b_i = x_1$. Since $z_i \in Z_i(K)$, we have $b_i \in (x_1 : x_2)$ for all $i$ and therefore $a_i\cdot b_i = \delta_i x_1$ for some $\delta_i \in R$, as $x_1, x_2$ form a weak sequence. Using $\Sigma a_i\cdot b_i = x_1$, we obtain that there is an $i_0$ such that $\delta_i \not\in \mathfrak{m}$, hence a unit in $R$. Consider $w := \delta_i^{-1}z_i = a_iT_1 - b_iT_2 \in Z_i(K)$, put $\alpha := \delta_i$, then

$$aw - (x_2T_1 - x_1T_2) = (aa - x_2) T_1 \in \mathfrak{m}Z_i(K).$$

Now $w = aT_1 - bT_2$ is a cycle, so $a \in (x_2 : x_1)$, hence $aa \in (x_2)$. So we may write $aw - (x_2T_1 - x_1T_2) = \mu x_2 T_1$ with some $\mu \in R$. Therefore $\mu x_2 = 0$, and by the weak sequence property we derive $\mu m^2 = 0$.

If $x_1 \not\in m^2$ or $x_2 \not\in m^2$, then certainly $z = x_2T_1 - x_1T_2 \in m^2 K \supseteq \mathfrak{m}Z_i(K)$ a contradiction to our assumption. Hence we may assume that $x_1, x_2 \in m^2$, so $x_2\mu \in m^2 \mu = 0$ and therefore $aw - (x_2T_1 - x_1T_2) = 0$. So $aa = x_2$ and $ab = x_1$.

As $b \in (x_1 : x_2)$, we have $mb \subseteq (x_1)$. Now consider the multiplication map $b : m \to (x_1)$ given by the multiplication with $b$. We have the exact sequence

$$0 \to \text{Ker}(b \cdot) \to m \to (x_1) \to 0.$$

Tensoring with $R/m$, we have a surjective map $m/m^2 \to (x_1)/m(x_1)$. Since $ab = x_1$, $a \in m \setminus m^2$. Let $m_1, \ldots, m_n$ denote a minimal system of generators of $m$ with $m_1 = \alpha$ and $m_i, b \in m(x_i)$ for $i = 2, \ldots, n$. Now write $m_i b = \gamma_i x_i$ with $\gamma_i \in m$ for $i = 2, \ldots, n$; then $\alpha \gamma_i - m_i \in \text{Ker}(b \cdot)$, hence $\alpha \gamma_i - m_i \in (0 : x_i) = (0 : m)$ for $i = 2, \ldots, n$.

So for all $i \geq 2$, $j \geq 1$ we have $m_j(\alpha \gamma_i - m_i) = 0$, hence $m_j m_i \in m^3 \cap (\alpha)$. Therefore $m^2/m^3$ is generated by the class of $m^2 = \alpha^2$, so $m^2$ is generated by $\alpha^2$ by Nakayama’s Lemma, hence $m^k$ is generated by $\alpha^k$ for all $k \geq 2$. So we may write $x_1 = u_1\alpha^{n_1}, x_2 = u_2\alpha^{n_2}$ with $u_1, u_2$ units in $R$ and $n_1, n_2$ natural numbers. Since $n_1 \leq n_2$ or $n_2 \leq n_1$ we have $(x_1, x_2) = (x_1)$ or $(x_1, x_2) = (x_2)$ the desired contradiction.

Proposition 1. Let $(R, m)$ be a local ring, $x_1, \ldots, x_n$ be an unconditioned weak sequence such that $(x_1, \ldots, x_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$, then the elements $d(T_i)$ for $I \subseteq \{1, \ldots, n\}$, $\#I \geq 2$ form part of a basis of $Z(K)/mZ(K)$, where $K = K(\mathfrak{z}, R)$.

Proof. Let $I \subseteq \{1, \ldots, n\}$ and denote

$$z = \sum \limits_{i \in I} (-1)^{\sigma(i, I)} x_i T_i,$$

where $\sigma(i, I) = \# \{ j \in I : j < i \}$, then we have to show $z \not\in \mathfrak{m}Z(K)$.

Let $R \to \bar{R} := R/(x_j)_{j \not\in I}$ be the canonical surjection, which induces a homomorphism $K(\mathfrak{z}, R) \to K(\mathfrak{z}, \bar{R})$. We reduce $K(\mathfrak{z}, \bar{R})$ modulo the ideal $\langle T_j \rangle_{j \not\in I}$ and obtain a homomorphism of differential graded algebras

$$K(x_1, \ldots, x_n; R) \to K((x_i)_{i \in I}; R/(x_j)_{j \not\in I}).$$
Now it is enough to show that the image of \( z \) under this homomorphism is not contained in \( \overline{m}K((x_i)_{i \in I}; \ R/(x_j)_{j \notin I}) \), where \( \overline{m} \) denotes the maximal ideal of \( R/(x_j)_{j \notin I} \).

So without loss of generality by renumbering we only have to show

\[
z = \sum_i (-1)^{i+1} x_i T_1 \cdot \ldots \cdot T_i \cdot \ldots \cdot T_n \notin \overline{m}Z_{n-1}(K).
\]

Let us assume this is not true and reduce again modulo \( x_3, \ldots, x_n \), then

\[
z = \tilde{x}_1 T_2 \cdot \ldots \cdot T_n - \tilde{x}_2 T_1 T_3 \cdot \ldots \cdot T_n \in \overline{m}Z_{n-1}(K(x, R/(x_3, \ldots, x_n))),
\]

hence there exist \( a_j \in m \), \( z_j \in Z_{n-1}(K(x, R/(x_3, \ldots, x_n))) \), \( z_j = \sum_i (-1)^{i+1} a_j x_i T_1 \cdot \ldots \cdot \hat{T}_i \cdot \ldots \cdot T_n \), such that \( z = \sum_j a_j z_j \). Since \( \delta z_j = 0 \), we have \( a_j x_j T_3 \cdot \ldots \cdot T_n - a_j x_1 T_3 \cdot \ldots \cdot T_n = 0 \), hence \( a_j x_2 - a_j x_1 = 0 \) for all \( j \) and \( \sum_j a_j a_{j+1} = \tilde{x}_1, \sum_j a_j a_{j+2} = \tilde{x}_2 \). So consider \( w_j := a_j T_1 - a_{j+2} T_2 \in Z_1(K(\tilde{x}_1, \tilde{x}_2; R/(x_3, \ldots, x_n))) \), then

\[
\sum_j a_j w_j = \tilde{x}_2 T_1 - \tilde{x}_1 T_2 \in \overline{m}Z(K(\tilde{x}_1, \tilde{x}_2; R/(x_3, \ldots, x_n))),
\]

which contradicts Lemma 2. This proves the assertion of the Proposition.

**Remark.** There exist unconditioned weak sequences \( \overline{x} \), for which the assumption \( (x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n) \) for all \( 1 \leq i \leq n \) does not hold and hence the assertion of the proposition cannot be true in these cases.

**Example.** Let \( (R, m) \) be a regular local ring of dimension 1 and let \( x \) be a generator of \( m \). Consider \( x_1 := x, x_2 := x^r \ (r \geq 1) \). Then \( 0 : x_1 = (0 : x_2) = 0, (x_1 : x_2) = R \). Hence \( m(x_1 : x_2) = (x_1), (x_2 : x_1) = (x^{r-1}) \), hence \( m(x_2 : x_1) = (x_2) \).

**Lemma 3.** Let \( (R, m) \) be a local ring, \( x_1, \ldots, x_n \) an unconditioned weak sequence such that \( (x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n) \) for all \( 1 \leq i \leq n \). Let \( I \subseteq \{1, \ldots, n\} \) and denote \( z = \sum_{i \in I} a_i z_i \in K(x, R) \) a cycle, then for every \( i \in I \) and for every \( \beta \in m \) we have \( \beta a_i \in m(x_i) \).

**Proof.** By a suitable renumbering we may assume \( I = \{1, \ldots, s\} \). Modulo the ideal \( (x_1, \ldots, x_s) \) we consider \( \tilde{z} = \tilde{a}_1 T_1 + \tilde{a}_2 T_2, \tilde{z} \) is a cycle in \( K(x, R/(x_3, \ldots, x_n)) \). Now it is enough to show that \( \beta a_1 \in m(x_2) \) and \( \beta a_2 \in m(x_1) \) modulo \( (x_3, \ldots, x_n) \) for every \( \beta \in m \). So without loss of generality \( n = 2, z = a T_1 - b T_2 \) and we show \( \beta a \in m(x_2), \beta b \in m(x_1) \) for \( \beta \in m \).

So take any \( \beta \in m \), then there exist \( u_1, u_2 \in R \) such that \( \beta a = u_2 x_2, \beta b = u_1 x_1 \) by the weak sequence property.

Now \( u_1, u_2 \) cannot simultaneously be units in \( R \). If so, we have \( u_1 x_1 x_2 = u_2 x_1 x_2, \) since \( z = a T_1 - b T_2 \) is a cycle, and hence by the weak sequence property \( (u_1 - u_2) x^m = 0 \). So \( \alpha = u_2^{-1} u_1 \alpha \) for every \( \alpha \in m^2 \). Now since \( x_1, x_2 \) is an unconditioned weak sequence, so is \( x_1', x_2' \), where we denote \( x_1' := u_1 x_1, x_2' := u_2 x_2 \). Consider the cycle \( z' := u_2 a T_1 - u_1 b T_2 \in K(x_1', x_2'; R) \). Then

\[
\beta z' = \beta a T_1 - \beta b T_2 = \beta a T_1 - \beta b T_2
\]

\[
= x_2' T_1 - x_1' T_2 \in mZ(K),
\]

which contradicts the assertion of Lemma 2.

So at least one of the coefficients \( u_1, u_2 \) is a nonunit. Let \( u_1 \in m \). We have to show that also \( u_2 \in m \). So assume \( u_2 \) is a unit, without loss of generality \( u_2 = 1 \). Then
unconditioned weak sequences

$x_1x_2 = \beta ax_1 = \beta bx_2 = u_1x_1x_2$, hence $x_1x_2 = 0$. We obtain $x_2 \in (0 : x_1) = (0 : m)$, therefore $x_2m = 0$ hence $(0 : x_2) = m$. But then we have $x_2 = \beta a \in m^2 = m(0 : x_2) = m(0 : m) = 0$ a contradiction.

COROLLARY. Let $(R, m)$ be a local ring, $x_1, \ldots, x_n$ an unconditioned weak sequence such that $(x_1, \ldots, x_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$, then $m((x_j)_{j \in I : x_j}) \subseteq m(x_j)_{j \in I : x_j}$.

PROOF. Let $a \in (x_j)_{j \in I : x_j}, \beta \in m$, then there exist $y_j \in R$ for $j \in I$ such that $ax_j = \Sigma_{j \in I : y_j}x_j$. Then $\alpha T_i = \Sigma_{j \in I : y_j}T_i$ is a cycle in $K(\xi, R)$, hence by Lemma 3 we have $\beta a \in m(x_j)_{j \in I : x_j}$.

We are now able to prove the following

THEOREM. Let $(R, m)$ be a local ring, $x_1, \ldots, x_n$ be a sequence of elements such that $(x_1, \ldots, x_i, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$, then the following are equivalent:

1) $x_1, \ldots, x_n$ is an unconditioned weak sequence,
2) for all $I \subseteq \{1, \ldots, n\}, 1 \leq k \leq \#I, J_0 \subseteq I$ with $\#J_0 = \#I - k$:

$$H_k(K(x_i)_{i \in I})/i\left(\left(m(\xi) \Sigma_{j \in k} K_k(x_j)_{j \in j} \right) \cap Z_k(K(x_i)_{i \in I})\right)$$

is a vector space,

3) for all $I \subseteq \{1, \ldots, n\}, i_0 \in I$:

$$H_i(K(x_i)_{i \in I})/i\left(\left(m(\xi)K_1(x_i)_{i \in I_{i_0}} \right) \cap Z(K(x_i)_{i \in I})\right)$$

is a vector space,

4) for all $I \subseteq \{1, \ldots, n\}, i_0 \in I$:

$$H_i(K(x_i)_{i \in I})/H_i(K(x_i)_{i \in I_{i_0}})$$

is a vector space.

($i$ denotes the canonical map $Z_\ast(K) \rightarrow H_\ast(K)$)

PROOF. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1). Consider the long exact homology sequence associated to

$$0 \rightarrow K(x_i)_{i \in I_{i_0}} \rightarrow K(x_i)_{i \in I} \rightarrow K(x_i)_{i \in I_{i_0}}[-1] \rightarrow 0,$$

then we obtain that the vector space $H_i(K(x_i)_{i \in I})/H_i(K(x_i)_{i \in I_{i_0}})$ is just the kernel of the multiplication map

$$R/(x_i)_{i \in I_{i_0}} \xrightarrow{\alpha} R/(x_i)_{i \in I_{i_0}},$$

hence $m((x_j)_{j \in I_{i_0}} : x_j) \subseteq (x_j)_{j \in I_{i_0}}$.

(1) $\Rightarrow$ (2). Let $\#I = r, 1 \leq k \leq r$ and $J_0 \subseteq I$ with $\#J_0 = r - k$. Consider

$$z = \sum_{\#G = k, R \subseteq I} \alpha R T_R \in Z_k(K(x_i)_{i \in I}).$$
Denote $S_0 := I \setminus J_0$, then $\alpha_{S_0} \in ((x_j)_{j \in S_0} : (x_j)_{j \in W})$, so for any $\beta \in m$: $\beta \alpha_{S_0} \in m(x_j)_{j \notin S_0}$ by the corollary of Lemma 3. So write

$$\beta \alpha_{S_0} = \sum_{j \notin W_0} \gamma_j x_j$$

with $\gamma_j \in m$.

hence

$$\beta \alpha_{S_0} = \sum_{j \notin W_0} \gamma_j T_{W_0} \in \left( m(\tau) \sum_{j \notin W_0} K(x_j)_{j \in \tilde{I}} \right) \cap Z(K(x_i)_{i \in \tilde{I}}),$$

since for each $S_0$ we also have $\beta \alpha_{S_0} \in m(\tau)$ again by the corollary of Lemma 3.

Remark 1. If $x_1, \ldots, x_n$ is an unconditioned relatively $m$ regular sequence with respect to $m(\tau)$ in the sense of Fiorentini [1], then $m(\tau)K(x_i)_{i \in \tilde{I}} \cap Z(K(x_i)_{i \in \tilde{I}}) \subseteq m B_1(K(x_i)_{i \in \tilde{I}})$ for all $I \subseteq \{1, \ldots, n\}$ by Fiorentini’s result, hence the submodules by which we have to divide $H_1(K(x_i)_{i \in \tilde{I}})$ in (2) and (3) in the theorem vanish.

For a system of parameters $\tau$ of a Buchsbaum ring this assumption certainly holds, since in this case $\tau$ is even a $d$-sequence (cf. [2]).

Remark 2. In general for unconditioned weak sequences the homology modules are not vector spaces.

Example. Let $A = k[[X_1, X_2, Y_1, Y_2, Z]]$ be the powerseries ring in five variables over a field $k$ and denote $m$ the maximal ideal of $A$.

Let

$$R := A/(Y_1^2, Y_1X_2, Y_2Y_2, Y_2X_1, Y_1m^2, Y_2m^2, Z^2, Zm^2, ZY_1, ZY_2 - Y_1X_2).$$

denote $m$ the maximal ideal of $R$ and $x_1, x_2, y_1, y_2, z$ the residues of the corresponding elements of $A$.

Then

(a) $x_1, x_2$ is an unconditioned weak sequence.

(b) $H_1(x_1, x_2; R)$ is not a vector space.

Proof. (a) $(0 : x_1) = (0 : x_2) = (y_1, y_2, z)m$,

$$(x_1 : x_2) = (y_1, y_2, z)m + (y_1, x_1),$$

$$(x_2 : x_1) = (y_1, y_2, z)m + (y_2, x_2).$$

Obviously $x_1, x_2$ is an unconditioned weak sequence.

(b) Consider $y_2T_1 - y_1T_2 \in Z(K(x_1, x_2; R))$. Then $0 \neq zy_2T_1 - zy_1T_2 = zy_2T_1 \in m Z(K)$. Notice that $R$ is a homogeneous ring. Assume now that $zy_2T_1 \in B(K)$, then there exists a linear form $f$ such that $\delta(fT_1T_2) = zy_2T_1$, but $\delta(fT_1T_2) = fx_1T_2 - fx_2T_1$, hence $f \in (0 : x_1) \subseteq m^2$ a contradiction.

Remark 3. The example in the remark to Proposition 1 shows for $r \geq 3$ that the theorem is not valid without the assumption $(x_1, \ldots, x_n) \subseteq (x_1, \ldots, x_n)$ for all $1 \leq i \leq n$. In this particular case $H_1(K(x_i)) = 0$ and $x^r \neq H_1(K(x_1, x_2)) \neq 0$.

Using the theorem and the following Proposition 2 we now would like to give a new proof of the characterization of Buchsbaum rings in terms of the Koszul homology. This characterization is already known using Suzuki [7] and Shimoda [3]:

The following are equivalent:

(1) $R$ is a Buchsbaum ring.

(2) for all systems of parameters $\tau$: $H_*(\tau, R)$ is a vector space,
(3) for all systems of parameters $\tau$: $H_1(\tau, R)$ is a vectorspace,
(4) for all systems of parameters $\tau$: $\tilde{H}_1(\tau, R)$ is a vectorspace,
(5) for all systems of parameters $\tau$: $\tilde{H}_1(\tau, R)$ is a vectorspace.

$\tilde{H}_1(\tau, M) := \text{coker}(\tilde{H}_1(\tau, (\tau)M) \to H_1(\tau, M))$ for a $R$-module $M$.

(1) $\Rightarrow$ (2) follows from our theorem and Remark 1. Obviously (2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5). We only need to show (5) $\Rightarrow$ (1), hence e.g. every system of parameters $\tau$ is a weak sequence. The following proposition which we prove more general for the module case (cf. Introduction) implies the desired result.

**Proposition 2.** Let $(R, m)$ be a local ring, $M$ a finitely generated $R$-module and $x_1, \ldots, x_n$ a sequence of elements in $m$ such that $H_1(x_1^{*1}, \ldots, x_n^{*n}; M)$ is a vectorspace for all multi-indices $(v_1, \ldots, v_n)$ with $v_i \geq 1$, then $x_1^{*1}, \ldots, x_n^{*n}$ is a weak sequence on $M$ for all $(v_1, \ldots, v_n)$.

**Proof.** It is only left to show that $\tau$ is a weak sequence. We will do this in two steps: (1) $m((x_1, \ldots, x_{n-1}; (\tau)M : x_n) \subseteq (x_1, \ldots, x_{n-1})M$. (2) If $H_1(x_1^{*1}, \ldots, x_n^{*n}; M)$ is a vectorspace for all $v_1, \ldots, v_n$, then $H_1(x_1^{*1}, \ldots, x_n^{*n}; M)$ is a vectorspace for all $v_1, \ldots, v_n$.

Then descending step by step we obtain the conclusion.

To (1). We have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & K(x_1, \ldots, x_{n-1}; (\tau)M) & \to & K(x_1, \ldots, x_n; (\tau)M) & \to & K(x_1, \ldots, x_{n-1}; (\tau)M)[-1] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K(x_1, \ldots, x_{n-1}; M) & \to & K(x_1, \ldots, x_n; M) & \to & K(x_1, \ldots, x_{n-1}; M)[-1] & \to & 0
\end{array}
\]

Taking homology we get the following commutative diagram:

\[
\begin{array}{ccccccccc}
\varphi_\tau & & \varphi_0 & & \varphi & & \varphi_0 & & \\
H_1(x_1, \ldots, x_n; (\tau)M) & \to & H_0(x_1, \ldots, x_{n-1}; (\tau)M) & \to & \text{coker } \varphi_\tau & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H_1(x_1, \ldots, x_n; M) & \to & H_0(x_1, \ldots, x_{n-1}; M) & \to & \text{coker } \varphi & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\tilde{H}_1(x_1, \ldots, x_n; M) & \to & \text{coker } \psi & \to & \text{coker } \psi_0 & & \\
\end{array}
\]

In other terms

\[
\begin{array}{ccccccccc}
H_1(\tau; (\tau)M) & \to & (\tau)M/((\tau)(x_1, \ldots, x_{n-1})M & \xrightarrow{\gamma} & (\tau)^2M/(\tau)(x_1, \ldots, x_{n-1})M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_1(\tau; M) & \to & M/(x_1, \ldots, x_{n-1})M & \xrightarrow{\gamma} & (\tau)M/(x_1, \ldots, x_{n-1})M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\tilde{H}_1(\tau; M) & \to & M/(\tau)M & \xrightarrow{\gamma} & (\tau)M/(x_1, \ldots, x_{n-1}, x_n^2)M & & \\
\end{array}
\]

By the snake lemma $\tilde{H}_1(\tau, M)$ maps onto the kernel of

\[
M/(\tau)M \xrightarrow{\gamma} (\tau)M/(x_1, \ldots, x_{n-1}, x_n^2)M.
\]
hence

\[ m \left( (x_1, \ldots, x_{n-1}, x_n^2)M : x_n \right) \subseteq (x_1, \ldots, x_n)M. \]

Now let \( \alpha \in (x_1, \ldots, x_{n-1})M : x_n \) we have to show that \( \alpha m \subseteq (x_1, \ldots, x_{n-1})M. \) But

\[ \alpha x_n \in (x_1, \ldots, x_{n-1})M \subseteq (x_1, \ldots, x_{n-1}, x_{n+1}^k)M \text{ for all } k \geq 1, \]

so \( \alpha x_n \in (x_1, \ldots, x_{n-1}, x_{n+1}^k)M. \) Now repeating the above argument for \( m \alpha \subseteq (x_1, \ldots, x_{n-1}, x_n^k)M \) for all \( k \geq 1, \) hence \( m \alpha \subseteq (x_1, \ldots, x_{n-1})M. \)

To (2). Again taking homology from (1) we derive Diagram A.

By the snake lemma we have an exact sequence

\[ K \rightarrow C(x_1, \ldots, x_n; M) \rightarrow \tilde{H}_1(\xi; M). \]

The map \( K \rightarrow C(x_1, \ldots, x_n; M) \) is actually the zero map. To prove this, consider \( \alpha \in M \) with \( \bar{\alpha} \in K, \) then \( \alpha \in (x_1, \ldots, x_{n-1})M. \) Let \( \alpha = \sum_{i=1}^{n-1} x_i \beta_i, \beta_i \in M \) then

\[ x_n \alpha = \sum_{i=1}^{n-1} x_n x_i \beta_i, \]

so \( z = \alpha T_n - \sum_{i=1}^{n-1} \beta_i x_i T_i \) is a cycle in \( K(\xi, (\xi)M) \) which maps onto \( \bar{\alpha}. \) But \( z = \sum_{i=1}^{n-1} \beta_i T_i T_n \) is a boundary in \( K_1(x_1, \ldots, x_n; M). \) So \( C(x_1, \ldots, x_n; M) \) is a vector-space since \( \tilde{H}_1(\xi; M) \) is.

Again the same argument is true for the sequence \( x_1, \ldots, x_{n-1}, x_n^k \) for all \( k \geq 1, \) hence for all \( k \geq 1: C(x_1, \ldots, x_{n-1}, x_n^k; M) \) is a vector-space. Therefore also

\[ C'(x_1, \ldots, x_{n-1}, x_n^k; M) \]

\[ := \text{coker} \left( H_1(x_1, \ldots, x_{n-1}; (x_1, \ldots, x_{n-1}, x_n^k)M) \rightarrow H_1(x_1, \ldots, x_{n-1}; M) \right) \]

is a vector-space.

Finally consider Diagram B.

By the snake lemma we have the exact sequence

\[ L \rightarrow \tilde{H}_1(x_1, \ldots, x_{n-1}; M) \rightarrow C'(x_1, \ldots, x_{n-1}, x_n^k; M). \]

So \( \tilde{H}_1(x_1, \ldots, x_{n-1}; M) \) modulo the submodule generated by all cycles \( z = \sum_{i=1}^{n-1} \alpha_i T_i \) with \( \alpha_i \in (x_1, \ldots, x_{n-1}, x_n^k)M \) is a vector-space, hence for every cycle \( z \in K_1 \) we have

\[ m z \subseteq \bigcap_{k \geq 1} (x_1, \ldots, x_{n-1}, x_n^k) K_1 = (x_1, \ldots, x_{n-1}) K_1, \]

so \( m \tilde{H}_1(x_1, \ldots, x_{n-1}; M) = 0. \)

We now want to look at the minimal \( R \)-free resolution of \( R/(x) \) for an unconditioned weak sequence \( \xi. \)

By Lemma 2 the Koszul complex is always a (module-) direct summand of this resolution for \( n \leq 2. \) But beside this fact in general very little can be said about what this resolution looks like. If \( \xi \) is just a sequence of one element \( x_1, \) then the resolution of \( R/(x_1) \) is given by

\[ \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow R \rightarrow R \rightarrow R/(x_1) \rightarrow 0, \]

where \( (F_*, d_*) \) is a direct sum of \( \dim_R R/(0 : m) = \dim R/m H_i(x_1) \) copies of the minimal \( R \)-resolution of the residue class field \( R/m \) and \( \varphi \) maps the basis elements of \( F_0 \) to representatives of a minimal system of generators of \( H_i(x_1). \) So in this case finding the \( R \)-resolution of \( R/(x_1) \) is equivalent to finding the \( R \)-resolution of \( R/m. \)
So one may ask in general: Assume we know the $R$-resolutions of the homology modules $H_i(\bar{\alpha})$, what can be said about the $R$-resolution of $R/(\bar{\alpha})$ in terms of these given resolutions?

We want to give an answer to this question in a very special case: namely $\bar{\alpha} = x_1, x_2$ is an unconditioned weak sequence of two elements and $\bar{\alpha}$ is unconditioned $m$-relatively regular with respect to $m(\bar{\alpha})$.

In this case $H_i(\bar{\alpha})$, $H_2(\bar{\alpha})$ are vector spaces by Remark 1 to the theorem. Denote $M, N$ free $R$-modules of rank $\dim H_i(\bar{\alpha})$ and $\dim H_2(\bar{\alpha})$ respectively and denote $(G, d)$ the resolution of the residue class field $R/m$. Below we will give an equivalent condition for the $R$-resolution of $R/(\bar{\alpha})$ to be isomorphic to

$$(\times) \quad K \oplus (G \otimes M[-2]) \oplus (G \otimes N[-3])$$

with obvious maps $G_0 \otimes M[-2] \to K_1$, $G_0 \otimes N[-3] \to K_2$ killing the homology $H_2(K)$.

Denote $\{z_j \mid j = 1, \ldots, s\}$ a set of cycles in $K_i(\bar{\alpha})$ which represent a $R/m$ vector space basis of $H_i(\bar{\alpha})$. Let $m_1, \ldots, m_s$ be a minimal system of generators of the maximal ideal $m$. For every pair $m_i, z_j$ we choose an element $y(m_i, z_j) \in m$ such that $\delta(y(m_i, z_j)T_1T_2) = m_i z_j$. This is possible, see Remark 1. Let $\mathfrak{A}$ be the $R$-ideal

$$\mathfrak{A} := \left( \sum_i r_i \gamma(m_i, z_j) / j = 1, \ldots, s; \sum_i r_i m_i = 0 \right).$$

Notice that the choices for $\gamma(m_i, z_j)$ are determined up to socle-elements, so as any relation $\sum r_i m_i = 0$ only involves $r_i \in m$ the ideal $\mathfrak{A}$ is well defined.

If $\sum r_i m_i = 0$ then

$$\sum_i r_i \gamma(m_i, z_j)(x_1T_1 - x_2T_2) = \delta\left( \sum_i r_i \gamma(m_i, z_j)T_1T_2 \right) = \left( \sum_i r_i m_i \right) z_j = 0.$$ 

Hence $\sum r_i \gamma(m_i, z_j) \in (0 : m)$, so we obtain $\mathfrak{A} \subseteq (0 : m)$.

Claim. The minimal $R$-resolution $(F, d)$ of $R/(\bar{\alpha})$ is isomorphic to $(\times)$ if and only if $\mathfrak{A} = 0$.

Proof. Clearly the $R$-resolution $F$ of $R/(\bar{\alpha})$ starts with

$$F_0 = R,$$

$$F_1 = RT_1 \oplus RT_2; T_1 \mapsto x_1, T_2 \mapsto x_2,$$

$$F_2 = RT_1T_2 \oplus G_0 \otimes M[2].$$

Denote $S_j, j = 1, \ldots, s$, a free basis of $M[-2]$ and let $1 \otimes S_j$ map to $z_j$ for $j = 1, \ldots, s$.

So we may define for $i = 1, \ldots, n$, $j = 1, \ldots, s$:

$$d(V_i \otimes S_j) = m_i (1 \otimes S_j) - \gamma(m_i, z_j)T_1T_2,$$

where $V_1, \ldots, V_n$ denote a $R$-basis of $G_1$ which is mapped to $m_1, \ldots, m_n$ in $G$.

If $\mathfrak{A} = 0$, then we can define

$$d \mid_{G \otimes M[-2]}, \tau_4 = (d_g)_{\tau_2} \otimes \text{id} \mid_{M[-2]},$$

$$d \mid_{G \otimes N[-3]}, \tau_4 = (d_g)_{\tau_1} \otimes \text{id} \mid_{N[-3]}$$

and $G_0 \otimes N[-3] \to K_2$ the obvious map killing the homology $H_2(K)$. So we have the resolution as desired.
If \( \mathfrak{A} \neq 0 \), then there is a relation \( \sum r_j m_j = 0 \), \( j \in \{1, \ldots, s\} \) with \( 0 \neq \sigma_j = \sum r_j \gamma(m_j, z_j) \in (0 : m) \). Hence the element \( \sigma_j T_1 T_2 \) is already killed by \( d(G_1 \otimes M[-2]) \) and therefore must not be killed by a basis element of \( G_0 \otimes N[-3] \). Also \( (\sum r_j m_j) \cdot (1 \otimes S_j) = 0 \), but \( (\sum r_j V_j) \otimes S_j \) is not a cycle in \( F \) although \( \sum r_j V_j \) is a cycle in \( G \). So in this case \((\times)\) even is not a complex. Hence the \( R \)-resolution of \( R/(\mathfrak{A}) \) is different from \((\times)\).

**References**


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