

PSEUDO ALGEBRAICALLY CLOSED FIELDS OVER RATIONAL FUNCTION FIELDS

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ABSTRACT. The following theorem is proved: Let T be an uncountable set of algebraically independent elements over a field K_0 . Then $K = K_0(T)$ is a Hilbertian field but the set of $\sigma \in G(K)$ for which $\tilde{K}(\sigma)$ is PAC is nonmeasurable.

Introduction. A field M is said to be *pseudo algebraically closed* (= PAC) if every nonempty absolutely irreducible variety V defined over M has an M -rational point.

If M is an algebraic extension of a field K and every absolutely irreducible polynomial $f \in K[X, Y]$, separable in Y , has infinitely many M -rational zeros, then M is PAC. This is a combination of Ax's application of "descent" [1] and the generic hyperplane intersection method as in Frey [3]. If $\sigma_1, \dots, \sigma_e$ are e -elements of the absolute Galois group $G(K)$ of K , then $\tilde{K}(\sigma)$ denotes the fixed field in \tilde{K} of $\sigma_1, \dots, \sigma_e$. Here \tilde{K} is the algebraic closure of K . We denote by μ the normalized Haar measure of $G(K)^e$. It is proved in [6, Lemma 2.4] that if K is a Hilbertian field, if $f \in K[X, Y]$ is an absolutely irreducible polynomial and if $A(f) = \{\sigma \in G(K)^e \mid f \text{ has a } \tilde{K}(\sigma)\text{-zero}\}$, then $\mu(A(f)) = 1$. If in addition K is countable, then there are only countably many f 's and therefore the intersection of all the $A(f)$'s is also a set of measure 1. Thus the set $S_e(K) = \{\sigma \in G(K)^e \mid \tilde{K}(\sigma) \text{ is PAC}\}$ has measure 1.

This basic result, which is called the *Nullstellensatz* in [6], has been the cornerstone for several model theoretic investigations of the fields $\tilde{K}(\sigma)$ (cf. [9, 7 and 4]).

If K is uncountable, then the above argument is not valid any more. It is our aim in this note to show that indeed the *Nullstellensatz* itself is not true in this case. More precisely, we prove

THEOREM. *Let T be an uncountable set of algebraically independent elements over a field K_0 . Then $K = K_0(T)$ is a Hilbertian field but $S_e(K)$ is a nonmeasurable subset of $G(K)^e$ for every positive integer e .*

1. The Haar measure of a profinite group. Let G be a profinite group and consider the boolean algebra of open-closed sets in G . They are finite unions of left cosets xN , where N are open normal subgroups. The σ -algebra generated by the open-closed sets is denoted by \mathfrak{B}_0 . Every open subset of G is a union of open-closed sets. We

Received by the editors February 5, 1982 and, in revised form, May 27, 1982.

1980 *Mathematics Subject Classification.* Primary 12F20.

Key words and phrases. PAC fields, Hilbertian fields, Haar measure.

¹Partially supported by the Fund for Basic Research administered by the Israel Academy of Sciences and Humanities.

denote by \mathfrak{B} the σ -algebra generated by the open sets. This is the *Borel*-algebra of G . Attached to \mathfrak{B} is the Haar measure μ of G . We make the convention that $\mu(G) = 1$ ensuring the uniqueness of μ . The σ -algebra generated by \mathfrak{B}_0 (resp. \mathfrak{B}) and the subsets of zero sets of \mathfrak{B}_0 (resp. \mathfrak{B}) is denoted by $\overline{\mathfrak{B}}_0$ (resp. $\overline{\mathfrak{B}}$). For every $B \in \overline{\mathfrak{B}}_0$ there exist $A, C \in \mathfrak{B}_0$ such that $A \subseteq B \subseteq C$ and $\mu(C - A) = 0$. The sets in $\overline{\mathfrak{B}}$ are the *measurable sets* of G .

LEMMA 1.1. *In the above notation we have $\overline{\mathfrak{B}}_0 = \overline{\mathfrak{B}}$.*

PROOF. It suffices to show that if U is an open set, then there exist $A, B \in \mathfrak{B}_0$ such that $A \subseteq U \subseteq B$ and $\mu(B - A) = 0$.

We write U as a union $U = \bigcup_{i \in I} x_i M_i$, where the M_i are open normal subgroups and $x_i \in G$, and let

$$\alpha = \sup \left\{ \mu \left(\bigcup_{i \in I'} x_i M_i \right) \mid I' \text{ is a countable subset of } I \right\}.$$

For every positive integer n there exists a countable subset J_n of I such that $\alpha - \mu(\bigcup_{i \in J_n} x_i M_i) < 1/n$. Then for $J = \bigcup_{n=1}^{\infty} J_n$, the set $A = \bigcup_{j \in J} x_j M_j \subseteq U$ belongs to \mathfrak{B}_0 and satisfies $\mu(A) = \alpha$.

Consider the closed normal subgroup $N = \bigcap_{j \in J} M_j$ of G . The corresponding quotient group G/N has a countable basis for its topology. Denote by $\pi: G \rightarrow G/N$ the canonical homomorphism. Then the sets $\pi(x_i M) = x_i M_i N/N$ are open in G/N and their union is πU . By a theorem of Lindelöf, I has a countable subset K such that $\pi U = \bigcup_{k \in K} \pi(x_k M_k)$. In addition $\pi^{-1}\pi A = A$ and $\pi(U - A) = \pi U - \pi A$, as one may easily check. Hence

$$(1) \quad U - A \subseteq \pi^{-1}\pi(U - A) = \bigcup_{k \in K} (x_k M_k N - A).$$

The right-hand side of (1), which we denote by B_0 , belongs to \mathfrak{B}_0 . If we prove that $\mu(B_0) = 0$, then $B = A \cup B_0$ is a set of \mathfrak{B}_0 that contains U and satisfies $\mu(B - A) = 0$. Of course, it suffices to prove that $\mu(x_k M_k N - A) = 0$ for every $k \in K$.

Assume that there exists a $k \in K$ such that $\mu(x_k M_k N - A) > 0$ and let n_1, \dots, n_r be representatives of N modulo $N \cap M_k$. Then $x_k M_k N - A = \bigcup_{\rho=1}^r (x_k M_k n_\rho - A)$ and therefore, there exists a $1 \leq \rho \leq r$ such that $\mu(x_k M_k n_\rho - A) > 0$. Note that $A n_\rho = A$, since $n_\rho \in M_j$ for every $j \in J$. Hence, $\mu(x_k M_k - A) = \mu((x_k M_k - A)n_\rho) = \mu(x_k M_k n_\rho - A) > 0$. It follows that

$$\mu \left(\bigcup_{j \in J} x_j M_j \cup x_k M_k \right) = \mu(A) + \mu(x_k M - A) > \alpha,$$

which contradicts the definition of α . \square

LEMMA 1.2. *In the above notation let S be a subset of G . Suppose that for every $B \in \mathfrak{B}_0$ there exists an epimorphism $r: G \rightarrow H$ such that (a) $B = r^{-1}rB$ and (b) $\mu_H(rS) = 1$. Then $G - S$ contains no subset of a positive measure. If also $G - S$ has the above property, then S is not measurable.*

PROOF. Assume that $G - S$ contains a set $\bar{B} \in \bar{\mathfrak{B}}$. By Lemma 1.1 there exists a set $B \in \mathfrak{B}_0$ such that $B \subseteq \bar{B}$ and $\mu(\bar{B} - B) = 0$. Let $r: G \rightarrow H$ be an epimorphism such that (a) and (b) hold. Then $\mu_H(r(G - B)) = 1$ and $G - B = r^{-1}r(G - B)$. It follows that $\mu(G - B) = 1$, hence $\mu(\bar{B}) = 0$. \square

2. Rational function fields of one variable. Let t be a transcendental element over an infinite field K and let $E = K(t)$. Then E is a Hilbertian field. If E is also countable, then, as noted in the introduction, $S_e(E) = \{\sigma \in G(K)^e \mid \tilde{E}(\sigma) \text{ is PAC}\}$ is a set of measure 1 for every $e \geq 1$. In the noncountable case we are able to prove only the following weaker result.

PROPOSITION 2.1. *If K is an uncountable field, then the complement of $S_e(E)$ in $G(E)^e$ contains no subsets of positive measure.*

The first step in the proof is a generalization of a basic result for polynomials in several variables. We use here both $\#A$ and $|A|$ to denote the cardinality of a set A .

LEMMA 2.2. *Let A be an infinite subset of a field K . If $F \subseteq K[X_1, \dots, X_n]$ is a set of nonzero polynomials and $|F| < |A|$, then $\#\{(a_1, \dots, a_n) \in A^n \mid f(a_1, \dots, a_n) \neq 0 \text{ for every } f \in F\} = |A|$.*

PROOF. Our assertion is true for $n = 1$, since every polynomial $f \in F$ has only finitely many zeros. Suppose, by induction, that the assertion is true for $n - 1$, where $n \geq 2$. Then, since every $f \in F$ has a nonzero coefficient $g \in K[X_1, \dots, X_{n-1}]$, we have $\#\{(a_1, \dots, a_{n-1}) \in A^{n-1} \mid f(a_1, \dots, a_{n-1}, X_n) \neq 0 \text{ for every } f \in F\} = |A|$. For every (a_1, \dots, a_{n-1}) in the above set there exists, by the case $n = 1$, an element $a_n \in A$ such that $f(a_1, \dots, a_n) \neq 0$ for every $f \in F$. Therefore, our assertion is also true for n . \square

COROLLARY 2.3. *If $\{U_i \mid i \in I\}$ is a family of nonempty Zariski K -open sets in A^n and $|I| < |A|$, then $|A^n \cap \bigcap_{i \in I} U_i| = |A|$.*

PROOF. Every U_i is defined by finitely many polynomial inequalities. \square

We define the *rank* of an infinite algebraic extension as the cardinality of the family of all finite subextensions. The *rank* of a finite algebraic extension is merely said to be *finite*.

A finite separable extension has only finitely many subextensions. Hence, if F is the compositum of m finite separable extensions of a field E and m is an infinite cardinal number, then $\text{rank}(F/E) = m$.

LEMMA 2.4. *Let F be a separable extension of E with $\text{rank}(F/E) < |K|$ and let $f \in E[X, Y]$ be an irreducible polynomial in $F[X, Y]$, separable in Y . Then there exists an $x \in E$ such that $f(x, y)$ is separable irreducible in $F[Y]$.*

PROOF. Let $\{E_i \mid i \in I\}$ be the family of all finite separable extensions of E which are contained in F . By assumption $|I| < |K|$. By a theorem of Inaba [5, §4], there exists for every $i \in I$ a nonempty Zariski K -open set $U_i \subseteq A^2$ such that if $(a, b) \in U_i(K)$, then $f(a + bt, y)$ is separable irreducible in $E_i[Y]$. The intersection

$\bigcap_{i \in I} U_i(K)$ is, by Corollary 2.3, not empty. If (a, b) lies in this intersection and $x = a + bt$, then $f(x, Y)$ is separable irreducible over every E_i , hence also over F . \square

LEMMA 2.5. *Let N be a Galois extension of E with $\text{rank}(N/E) < |K|$. Then every $\sigma_1, \dots, \sigma_e \in \mathcal{G}(N/E)$ can be extended to elements $\tau_1, \dots, \tau_e \in G(E)$, respectively, such that $\tilde{E}(\tau)$ is a PAC field.*

PROOF. We well-order the absolutely irreducible polynomials of $K[X, Y]$ which are separable in Y in a transfinite sequence $\{f_\alpha \mid \alpha < m\}$, where $m = |K|$, such that each of the polynomials appears \aleph_0 times in the sequence. For every $\alpha < m$ we define a finite separable extension E_α of E in which f_α has a zero and such that the set of fields $\{N\} \cup \{E_\alpha \mid \alpha < m\}$ is linearly disjoint over E .

Indeed let $\beta < m$ and assume, by transfinite induction, that E_α has been defined for every $\alpha < \beta$. Let F be the compositum of N and all the fields E_α with $\alpha < \beta$. Then F is a separable extension of E with $\text{rank}(F/E) < m$. By Lemma 2.4 there exists an $x \in E$ such that $f(x, y)$ is separable irreducible in $F[Y]$. If $y \in \tilde{E}$ satisfies $f(x, y) = 0$, then we may define $E_\beta = E(y)$ and E_β is linearly disjoint from F over E .

The compositum M of all the fields E_α is a separable algebraic extension of E which is linearly disjoint from N and which is PAC. The automorphisms $\sigma_1, \dots, \sigma_e$ may be extended to automorphisms $\tau_1, \dots, \tau_e \in G(M)$. Their fixed field $\tilde{K}(\tau)$ is an algebraic extension of M and hence is a PAC field itself. \square

PROOF OF PROPOSITION 2.1. We follow the pattern of Lemma 1.2 and note first that every open-closed set of $G(E)^e$ is determined by a finite Galois extension of E . It follows that if B is a set belonging to the σ -algebra \mathcal{B}_0 of $G(E)^e$ generated by the open-closed sets, then there exists a Galois extension N of E with $\text{rank}(N/E) \leq \aleph_0$ such that $r^{-1}rB = B$, where $r: G(E)^e \rightarrow \mathcal{G}(N/E)^e$ is the restriction map.

By Lemma 2.5, $rS_e \in \mathcal{G}(N/E)^e$. Hence, by Lemma 2.1, $G(E)^e - S_e(E)$ contains no sets of a positive measure.

3. Rational function fields of many variables. There is one case where we have enough information about the set $G(E)^e - S_e(E)$, which allows us to reach a decisive conclusion about the nonmeasurability of the set $S_e(E)$. This is the case where K itself is a rational function field of uncountably many variables over a field K_0 .

LEMMA 3.1. *Let T be a nonempty set of algebraically independent elements over a field L and let $M = L(T)$. Then every e elements $\sigma_1, \dots, \sigma_e$ of $G(L)$ can be extended to e elements ρ_1, \dots, ρ_e of $G(M)$ such that $\tilde{M}(\rho)$ is not a PAC field.*

PROOF. We single out an element $t \in T$ and denote $L' = L(T - \{t\})$. Then $\sigma_1, \dots, \sigma_e$ can be extended to elements $\sigma'_1, \dots, \sigma'_e$ of $G(L')$. We may therefore assume without loss that T consists of one element t .

Consider first the case where one of the σ_i 's is not the identity automorphism and note that L is algebraically closed in the field $F = L((t))$ of formal power series in t . Therefore, $\sigma_1, \dots, \sigma_e$ may be extended to elements $\hat{\rho}_1, \dots, \hat{\rho}_e$ of $G(F)$. The restrictions

ρ_1, \dots, ρ_e of $\hat{\rho}_1, \dots, \hat{\rho}_e$ to \tilde{M} are elements of $G(M)$ that extend $\sigma_1, \dots, \sigma_e$ and $\tilde{M}(\rho)$ is not a PAC field. Indeed, $\tilde{M}(\rho) = \tilde{M} \cap \tilde{F}(\hat{\rho})$ and $\tilde{F}(\hat{\rho})$ is a Henselian field with respect to a real-valued valuation defined by the specialization $t \rightarrow 0$. Therefore, $\tilde{M}(\rho)$ itself is Henselian (cf. Ax [2, Proposition 12]) and it is not separably closed. Theorem 2 of Frey [3] implies that $\tilde{M}(\rho)$ is not a PAC field.

If $\sigma_1 = \dots = \sigma_e = 1$, then noting that the separable closure M_s of M is not contained in $L_s((t))$, we may choose $\hat{\rho}_1, \dots, \hat{\rho}_e$ in $G(L_s((t)))$ that do not fix M_s . Then we proceed as before. \square

We are now in a position to prove our main result.

THEOREM 3.2. *Let T be an uncountable set of algebraically independent elements over a field K_0 and let $E = K_0(T)$. Then for every positive integer e , both $S_e(E)$ and $G(E)^e - S_e(E)$ contain no sets of positive measure. In particular $S_e(E)$ is nonmeasurable.*

PROOF. By Proposition 2.1 we have only to prove that $S_e(E)$ contains no sets of positive measure. Indeed, if $B \subseteq G(E)^e$ is open-closed, then there exists a finite subset T_0 of T and there exists a finite Galois extension F_0 of $K_0(T_0)$ such that B is the listing to $G(E)^e$ of a certain subset of $\mathcal{G}(F_0/K_0(T_0))^e$. It follows that if $B \in \mathfrak{B}_0$, then there exists a countable subset T_1 of T such that with $L = K_0(T_1)$ and $r: G(E)^e \rightarrow G(L)^e$ the restriction map, we have $B = r^{-1}rB$.

Note now that $E = L(T - T_1)$ and that $T - T_1$ is a nonempty set of algebraically independent elements over L . Hence, by Lemma 3.1, $r(G(E)^e - S_e(E)) = G(L)^e$. It follows by Lemma 1.2, that $S_e(E)$ contains no sets of positive measure. \square

COROLLARY 3.3. *If F is a finite extension of E , then $S_e(F)$ is a nonmeasurable set.*

PROOF. If $S_e(F)$ were measurable, then either $S_e(F)$ or its complement would have a positive measure in $G(E)^e$, a contradiction. \square

Note that there exist Hilbertian fields E_0 which are PAC (see [8, Theorem 3.3]). Every nonprincipal ultrapower E of E_0 is an uncountable Hilbertian PAC field. As already noted before every algebraic extension of E is again a PAC field. Hence, $S_e(E) = G(E)^e$ for every positive integer e . Thus Theorem 3.2 cannot be extended to arbitrarily uncountable Hilbertian fields.

The most interesting case which remains open is that of $E = \mathbb{C}(t)$.

PROBLEM. Are the sets $S_e((t))$ measurable?

Note that in case of a positive answer, we have $\mu(S_e(\mathbb{C}(t))) = 1$, by Proposition 2.1.

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